Macroeconomic Theory II

Miguel de Faria e Castro

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Preface

Based on material taught by Professor Ricardo Lagos on the third quarter of the Macroeconomic Theory sequence at New York University (Spring 2012).
All errors are my own.

Chapter 1

Asset Pricing

1.1 Introduction

While mainly associated with finance, asset pricing is of prime interest to macroeconomists. In fact, most of the seminal work done in this field was undertaken within the scope of macroeconomic models. This section reviews some of the most influential work in asset pricing, with emphasis on modeling issues and the main insights that are generated by the models.

Economic modeling, in any environment (not only in asset pricing) requires a careful definition of the **environment** or **primitives** on which agents will act. The key ingredients of a well-described environment are the following:

1. **Time** - Can be continuous or discrete. The time horizon can be finite or infinite.
2. **Population/Demographics** - Specifies what types of agents there are, if the population is countable or a continuum, and whether these agents are infinitely or finitely lived.
3. **Commodity Space/Endowments** - Specifies what agents can consume, and what can they trade to obtain consumption goods/services.
4. **Technology** - Specifies what can the economy produce, and with what types of inputs.
5. **Preferences** - How do agents rank/value each of the available commodities.
6. **Information Structure** - Whether information is complete, perfect or neither. Whether there is private information or informational asymmetries.
7. **Market Structure** - Specifies what can be traded and what cannot be traded. What assets are there? Are markets complete? Who owns firms? How many markets there are?

8. **Equilibrium Concepts** - These summarise the way agents interact with each other. In a competitive equilibrium, for example, agents only interact through the market, only adjusting their behviour and reacting to prices that are common knowledge across all agents [1].

1.2 Lucas 1978 - Asset Prices in an Exchange Economy

This is a seminal article on the asset pricing literature, having introduced for the first time, or pushed towards popular usage, several conceptual and methodological novelties: stochastic discount factors, the concept of rational expectations equilibrium, and rational pricing functions, among others. The article was also a testing ground for several well-known results in dynamic programming that would become widely used in the literature.

1.2.1 Environment

The basic environment is as follows (using the 'checklist' presented in the previous section):

1. Time is discrete and infinite.
2. There is a unit measure of *ex-ante* equal agents who are infinitely lived. We shall simplify this assumption and assume existence of a representative agent (RA) [2].
3. There is one durable good: a set of 'trees'. There are n different types of trees, each in unit measure. Every period t, each tree yields some dividend d_{it} (the 'fruit', $i \in \{1, \ldots, n\}$ indexes the type of tree). Let $d_t = (d_{1t}, \ldots, d_{nt})$ denote the vector of fruit yielded by the n trees at time t.
4. Fruit, the consumption good, is perishable. Let c_t denote the amount of fruit consumed by the RA at time t.
5. Feasibility imposes that, at each period, consumption should not exceed the total amount of fruit that was generated by the trees. That is

$$c_t \leq \sum_{i=1}^{n} d_{it}$$

[1] For example, when talking about *Competitive Equilibrium*, we are effectively applying the game-theoretic concept of Nash Equilibrium.

[2] The whole point of this, popularised as the *Lucas trick* for pricing assets, is that we are interested in the price variables, and not in studying quantities. This framework allows us to focus on equilibrium prices, while keeping quantities 'fixed'.

Under monotonic preferences, this constraint will be satisfied in equality (as the fruit is perishable, so it is wasted if not eaten today), thus feasibility is equivalent to market clearing for goods/fruit.

6. Preferences are defined over infinite sequences of consumption $c = \{c_t\}_{t=0}^{\infty}$. They are summarised by an utility function $U(c)$, which is time and state-separable with von-Neumann-Morgenstern utility $u(c_t)$:

$$U(c) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

The discount factor is $\beta \in (0, 1)$.

7. The information structure is such that there is aggregate uncertainty regarding the realisation of each tree's dividends.

8. In terms of endowments, each agent is born at $t = 0$ endowed with a tree of type i. In the context of the RA model, he owns all the trees in the economy.

9. There are two markets in this economy: a spot market for fruit, and a financial market where equity shares are traded. Clearly, in the RA formulation, both markets will clear trivially (the asset markets clearing at zero trades).

10. The Equilibrium concept is the usual Walrasian notion of Competitive Equilibrium.

1.2.2 The Individual Optimisation Problem

Any agent is born at $t = 0$ and chooses a sequence of consumption c_t and stock holdings s_{t+1} (which is a $n \times 1$ vector, given that there are n trees). The problem can be described as

$$\max_{\{c_t, s_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1.1}$$

subject to a sequence of one-period budget constraints

$$c_t + \phi_t s_{t+1} = (\phi_t + d_t) s_t$$

where ϕ_t is a $n \times 1$ vector of asset prices, and non-negativity constraints

$$c_t, s_{t+1} \geq 0$$

Note that $s_{t+1} \geq 0$ imposes no short-selling of tree equity. We shall start by imposing some structure on the general problem.

Assumption 1.2.1. *Tree dividends follow a* ***Markov Process*** *(not necessarily a Markov Chain). Thus the evolution of $\{d_t\}_{t=0}^{\infty}$ can be described as*

$$\mathbb{P}(d_{t+1} \leq x' | d_t = x) = F(x, x')$$

where $F : \mathbb{R}^{2n} \to [0, 1]$ is a transition function.

Note that the deep parameters of the economy are $\langle u, \beta, F \rangle$: preferences and the transition function. We shall impose the following assumptions on those:

Assumption 1.2.2. *$u : \mathbb{R}_+ \to \mathbb{R}$ is bounded, continuously differentiable, increasing, strictly concave and $u(0) = 0$.*

Assumption 1.2.3. *The transition function $F : \mathbb{R}^{2n}_+ \to \mathbb{R}$ (hidden assumption: all dividends are non-negative) is continuous, has a stationary distribution π such that*

$$\pi(y') = \int F(y', y) d\pi(y)$$

and for any continuous function $g : \mathbb{R}^n \to \mathbb{R}$, we have that $\int g(y')dF(y', y)$ is continuous in y as well.

This last assumption tells us that the conditional expectation of any continuous function (of today's dividends) will also be continuous on today's dividends. That is, not only is the transition function continuous, but it also preserves continuity of function under the conditional expectation operator. This property will be extremely important in the study and characterisation of the equilibrium pricing functions.

Why do we care about **pricing functions** at all? Clearly, from the agent's problem, it follows that the optimal choice of asset holdings (and, hence, of consumption) will depend on an Euler Equation, which in turn requires the individual to form expectations not only about tomorrow's dividends, but also about tomorrow's prices. We can easily see that, assuming an interior solution:

$$u'(c_t)\phi_t = \beta \mathbb{E}_t[(\phi_{t+1} + d_{t+1})u'(c_{t+1})]$$

Thus knowing next period's price will be fundamental for the individual to be able to optimally allocate consumption intertemporally. Thus the individual will have to form, in some way, expectations of the pricing function. It turns out that, in a stationary environment (as ours will be), we will be able to define a pricing function $\phi_t = \phi(d_t)$ that defines a map from dividends to prices. Therefore, endowed with the knowledge of the process for dividends, the agent is then able to form well-defined expectations for prices. This will be at the core of the concept of Rational Expectations Equilibrium. For now, let us present the typical definition of a competitive equilibrium:

Definition 1.2.1. *A* ***Competitive Equilibrium*** *is an allocation $\{c_t, s_{t+1}\}_{t=0}^{\infty}$, and a price system $\{\phi_t\}_{t=0}^{\infty}$ such that*

1. *The allocation solves the individual problem in 1.1, given prices.*
2. *Markets clear*

$$c_t = \sum_{i=1}^{n} d_{it}$$
$$s_t = \iota$$

for all $t \geq 0$.

Note that, in our case, given that we are dealing with a RA problem, market clearing will be trivial: the RA will simply consume all fruit yielded by trees and will not trade assets. This allows us to focus on prices and their properties only. This is an application of the well-known *Lucas' trick for pricing assets* (whose name comes from this article).

1.2.3 The Individual Problem in Recursive Form

Given that the problem at hand is stationary (that is, it does not matter, given endowments, whether the agent is solving it at t or at $s > t$, as the optimal policies will be of the same form), we can study it in its recursive formulation. In this case, there will be $2n$ state variables: n endogenous states (asset holdings chosen last period, s) and n exogenous states (dividends this period, d). The controls will be (c, s'), consumption and asset holdings to be carried on to the next period. The Bellman Equation is

$$V(s,d) = \max_{c,s'}\{u(c) + \beta \mathbb{E}_{d'}[V(s',d')|d]\} \tag{BE}$$

$$\begin{aligned} &\text{s.t.} \\ c + \phi(d)s' &= [\phi(d) + d]s \\ c &\geq 0 \\ s' &\geq 0 \end{aligned}$$

where, note, the expectation is defined as

$$\mathbb{E}_{d'}[V(s',d')|d] = \int V(s',d')dF(d',d)$$

1.2.4 Recursive Competitive Equilibrium

This formulation allows us to take advantage of a different equilibrium concept, which is well-suited to recursive problems:

Definition 1.2.2. *A* ***Recursive Competitive Equilibrium*** *is a collection of functions:*

- *A continuous and bounded value function* $V(s,d)$
- *Decisions rules* $h^c(s,d) : \mathbb{R}^{2n}_+ \to \mathbb{R}_+$ *and* $h^s(s,d) : \mathbb{R}^{2n}_+ \to \mathbb{R}^n_+$
- *A price function* $\phi : \mathbb{R}^n_+ \to \mathbb{R}^n$

such that

1. *Given* ϕ, $V(s,d)$ *solves (BE), and* (h^c, h^s) *are the associated optimal decision rules implied by (BE).*

2. Markets clear

$$h^c(s,d) = \sum_{i=1}^{n} d_i$$
$$h^s(s,d) = \iota$$

for all (s,d).

Existence of RCE requires, first and foremost, existence of all the objects that are listed: a value function and policy functions that solve (BE), as well as a pricing function. Characterisation of the value and policy functions is achieved through the use of standard dynamic programming arguments, as long as we allow ϕ to exist for a second. The following propositions establish existence (and uniqueness) of these objects.

Proposition 1.2.1. *For each continuous* ϕ*, there is a unique, bounded, continuous and non-negative* $V(s,d;\phi)$ *satisfying (BE). Moreover, for each* d*,* $V(s,d;\phi)$ *is increasing and concave in* s*.*

Proof. Start by defining the operator $T : CB(X) \to CB(X)$, a self-map on the space of bounded and continuous functions endowed with the sup norm. Let $X = \mathbb{R}_+^{2n}$. We can write the problem in (BE) for an arbitrary value function $f(x) = f(s,d)$ (where the price function ϕ is implicitly taken as a parameter) as

$$Tf(s,d) = \max_{s'}\{u[(\phi(d)+d)s - s'] + \beta \int f(s',d')dF(d',d)\}$$

subject to the constraint correspondence $\Gamma(s,d) = \{s' \in \mathbb{R}_+^n : s' \leq (\phi(d)+d)s\}$. We start by claiming that T is indeed a **self-map** on $(CB(X), \|\cdot\|_\infty)$. To see this, let $f(s,d)$ be bounded. Utility is, by assumption, bounded. Furthermore, the expectation of a bounded function must be bounded as well. Hence it follows that $Tf(s,d)$ is bounded. Now, if $f(s,d)$ is assumed to be continuous, do we also get continuity of $Tf(s,d)$? This follows from the Maximum Theorem: given our assumption that the conditional expectation preserves continuity, and given that we assumed continuous utility, we know that the argument of the maximisation problem is continuous. Furthermore, the constraint correspondence Γ is compact and continuous. Hence it follows that $Tf(s,d)$ will, indeed, be continuous.
This establishes T as a self-map. We now want it to be a **contraction mapping** on $CB(X)$. For this, we can use Blackwell's Sufficient Conditions:

1. (Monotonicity) For any pair of functions $f, g \in CB(X)$, we have that $f \geq g \Rightarrow Tf \geq Tg$. To see this, assume that $f \geq g$. Let s'_g denote the optimal portfolio

choice for the function g. Then

$$\begin{aligned} Tf(s,d) &= \max_{s'}\{u[(\phi(d)+d)s-s'] + \beta\int f(s',d')dF(d',d)\} \\ &\geq u[(\phi(d)+d)s-s'_g] + \beta\int f(s'_g,d')dF(d',d) \\ &\geq u[(\phi(d)+d)s-s'_g] + \beta\int g(s'_g,d')dF(d',d) \\ &= Tg(s,d) \end{aligned}$$

where the second line follows from the fact that s'_g is not necessarily optimal at f and the third from $f \geq g$.

2. (Discounting) For any $f \in CB(X)$ and $a \in \mathbb{R}_+$, we must have $T(f+a) \leq Tf + \beta a$ for some $\beta \in (0,1)$. Let a be some positive constant, then:

$$\begin{aligned} T(f+a)(s,d) &= \max_{s'}\{u[(\phi(d)+d)s-s'] + \beta\int f(s',d')dF(d',d) + \beta a\} \\ &= Tf(s,d) + \beta a \end{aligned}$$

with $\beta \in (0,1)$ by assumption.

Thus T satisfies all of Blackwell's conditions and is established as a contraction mapping on $CB(X)$.
Now, $CB(X)$ with the sup norm is a complete metric space. Therefore, by Banach's Fixed Point Theorem, there exists a unique fixed point for T, our value function $V(s,d)$. This establishes existence, uniqueness, boundedness and continuity of the value function. To see that the value function is non-negative, note that the constraint correspondence imposes $(\phi(d)+d)s - s' \geq 0$. By assumption, $u(0) = 0$ and u is increasing. Therefore, $u[(\phi(d)+d)s-s'] \in [0, u[(\phi(d)+d)s]]$. This means that

$$0 \leq V(s,d) \leq \frac{u[(\phi(d)+d)s]}{1-\beta}$$

Thus $V(s,d)$is non-negative (this also proves boundedness, as $\beta < 1$ and u is assumed to be bounded).

It remains to show that for each d, $V(s,d)$ is increasing and concave in s. This is done by applying the usual corollary of the Contraction Mapping Theorem. Let $s_1 > s_0$. If we show that $Tf(s,d) \in S \subseteq CB(X)$ and S is closed, for a generic $f \in S \subseteq CB(X)$, then the corollary implies that the fixed point will belong to S. This allows us to take advantage of the fact that the set of weakly increasing functions is closed. Furthermore, if $Tf(s,d) \in S' \subseteq S \subseteq CB(X)$ for some S' not necessarily closed (i.e. the set of strictly increasing functions), the corollary also establishes that the fixed point will belong in S'. Take any $f(s,d)$ weakly increasing and $s_1 > s_0$. Start by noting that the constraint correspondence is *monotonic*, in the sense that since $\phi(d) + d \geq 0$, we have that $x \in$

$\Gamma(s_0, d) \Rightarrow x \in \Gamma(s_1, d)$. Now, denote by s'_1, s'_0 the optimal choices at states s_1 and s_0, respectively:

$$\begin{aligned}
Tf(s_1, d) &= \max_{s' \in \Gamma(s_1,d)} \{u[(\phi(d) + d)s_1 - s'] + \beta \int f(s', d')dF(d', d)\} \\
&\geq u[(\phi(d) + d)s_1 - s'_0] + \beta \int f(s'_0, d')dF(d', d)\} \\
&\geq u[(\phi(d) + d)s_0 - s'_0] + \beta \int f(s'_0, d')dF(d', d)\} \\
&= Tf(s_0, d)
\end{aligned}$$

where the third line follows from monotonicity of u. If we further assume that u is strictly increasing, we get that $Tf(s, d)$ is strictly increasing in s. By the corollary, this is sufficient to show that the fixed point will be increasing in s. Also note that the second line follows from monotonicity of $\Gamma(s, d)$, as s'_0 is available if the state is s_1.

For concavity, let $s_1 > s_0$ and let $s_\lambda = \lambda s_1 + (1 - \lambda)s_0, \forall \lambda \in [0, 1]$. Let s'_1, s'_0 denote the optimal policies at s_1, s_0, respectively and let

$$s'_\lambda = \lambda s'_1 + (1 - \lambda)s'_0$$

We now take advantage of *convexity* of the constraint correspondence: if $s'_1 \in \Gamma(s_1, d)$ and $s'_0 \in \Gamma(s_0, d)$, then $\lambda s'_1 + (1-\lambda)s'_0 = s'_\lambda \in \Gamma(s_\lambda, d)$ (this is easy to check). Note further that s'_λ is NOT necessarily the optimal policy at s_λ. Note then that, for any concave $f \in CB(X)$:

$$\begin{aligned}
Tf(s_\lambda, d) &= \max_{s' \in \Gamma(s_\lambda,d)} \{u[(\phi(d) + d)s_\lambda - s'] + \beta \int f(s', d')dF(d', d)\} \\
&\geq u[(\phi(d) + d)s_\lambda - s'_\lambda] + \beta \int f(s'_\lambda, d')dF(d', d) \\
&= u[\lambda((\phi(d) + d)s_1 - s'_1) + (1 - \lambda)((\phi(d) + d)s_0 - s'_0)] + \beta \int f(\lambda s'_1 + (1 - \lambda)s'_0, d')dF(d', d) \\
&> \lambda u[(\phi(d) + d)s_1 - s'_1] + (1 - \lambda)u[(\phi(d) + d)s_0 - s'_0] \\
&+ \lambda\beta \int f(s'_1, d')dF(d', d) + (1 - \lambda)\beta \int f(s'_0, d')dF(d', d) \\
&= \lambda Tf(s_1, d) + (1 - \lambda)Tf(s_0, d)
\end{aligned}$$

Thus establishing concavity of Tf. Note that from the assumption that u is strictly concave, we get in fact strict concavity of the contraction operator and, hence of the fixed point (even if f is not strictly concave, but only weakly so - the same happens with monotonicity). $\square$

The following result also ensures that the value function is well-behaved

Proposition 1.2.2. *(Benveniste-Scheinkman) If $V(s, d; \phi)$ is attained at (c, s') with $c > 0$, then V is differentiable with respect to s at (s, d) and*

$$\frac{\partial V(s, d; \phi)}{\partial s_i} = u'(c)[\phi_i(d) + d_i]$$

Proof. (Informal Discussion) The proof of the general result for differentiability of a value function in the context of dynamic programming may be found in Stokey, Lucas & Prescott (Ch. 4). It can be easily checked that all assumptions for differentiability are satisfied if the value function is concave and the solution is interior (here we only require interior consumption, interior asset holdings will be trivially implied from market clearing). The expression for the derivative follows from the Envelope Condition associated with (BE), by taking the derivative of the value function with respect to s_i. □

1.2.5 Computing the Price Function

Assuming an interior solution, the first order condition with respect to s_i' from (BE) is

$$u'(c)\phi_i(d) = \beta \mathbb{E}_{d'} \left[\frac{\partial V(s', d'; \phi)}{\partial s_i'} | d \right]$$

Using the envelope condition and plugging in the above FOC gives us the Euler Equation for each asset i (we will have n of them)

$$u'(c)\phi_i(d) = \beta \mathbb{E}_{d'} \left[[\phi_i(d') + d_i'] u'(c') | d \right]$$

In equilibrium, the goods market clears at every period

$$c = \sum_{i=1}^{n} d_i$$
$$c' = \sum_{i=1}^{n} d_i'$$

Thus we can eliminate consumption from the Euler Equation:

$$u'\left(\sum_{i=1}^{n} d_i\right) \phi_i(d) = \beta \mathbb{E}_{d'} \left[[\phi_i(d') + d_i'] u' \left(\sum_{i=1}^{n} d_i'\right) | d \right]$$

This brings us closer to our main objective, which is to compute and characterise ϕ. In fact the n Euler Equations form a system of functional equations, which can, in principle, be solved for the n pricing functions (one for each asset). As we will see, dynamic programming methods will prove to be extremely useful in this endeavour: define two auxiliary functions (for each asset):

$$f_i(d) := u'\left(\sum_{i=1}^{n} d_i\right) \phi_i(d)$$
$$g_i(d) := \beta \int u'\left(\sum_{i=1}^{n} d_i'\right) d_i' dF(d', d)$$

f_i can be seen as the price of an asset in terms of marginal utility: how much utility does the consumer forego to acquire the asset today. g_i, on the other hand, represents the expected and discounted gain from acquiring the asset in terms of utility (which is marginal utility times the dividend), isolating for capital gains (possible gains from a rise in price that allows the asset to be sold at a higher price). The Euler Equation can then be rewritten as

$$u'\left(\sum_{i=1}^{n} d_i\right)\phi_i(d) = \beta \mathbb{E}_{d'}\left[\phi_i(d')u'\left(\sum_{i=1}^{n} d_i'\right)|d\right] + \beta \mathbb{E}_{d'}\left[d_i' u'\left(\sum_{i=1}^{n} d_i'\right)|d\right]$$

or, using our new functions

$$f_i(d) = g_i(d) + \beta \int f_i(d')dF(d', d) \tag{EE}$$

Note how useful it was to define the two functions: we can now treat the dividend gain as a separate payoff, while the price (which is what we want to get to) is treated as the argument of what appears to be conspicuously similar to a typical dynamic programming functional equation. Also, the above equation is economically meaningful: the price of the asset today (weighted by utility) should equal the value of the dividend that the asset will generate next period plus the gains from the right to resell the asset next period. That is, the price should reflect fundamental (dividend) and capital gains. Once we obtain the function $f_i(d)$, it is straightforward to retrieve the price of the asset as

$$\phi_i(d) = \frac{f_i(d)}{u'\left(\sum_{i=1}^{n} d_i\right)}$$

So, the only thing standing between us and the pricing function is whether or not it is possible to retrieve $f_i(d)$ from the above functional equation. If we could define $f_i(d)$ as the fixed point of a contraction mapping on the space of continuous and bounded functions, all would be well. The problem is that this is not immediate: we have no idea whether $g_i(d)$ is bounded (boundedness of instantaneous payoffs is crucial in establishing that T is a self-map on $CB(X)$). It is continuous by assumption: continuous differentiability of u implies continuous marginal utility, and the conditional expectation of a continuous function with respect to F is continuous, by assumption. It remains then to check whether $g_i(d)$ is bounded or not.

Lemma 1.2.1. *If $u'' \leq 0$, $u(0) = 0$ and $u(c) \leq B \in \mathbb{R}$, then (EE) has a unique, continuous and bounded solution.*

Proof. All assumptions to the lemma follow from the assumptions we imposed earlier on the primitives. From the fact that u is concave, we have that

$$u(x) \leq u(c) + u'(c)(x - c)$$

for any x. In particular, for $x = 0$ we get that

$$0 = u(0) \leq u(c) - cu'(c) \leq B - cu'(c) \Rightarrow cu'(c) \leq B$$

Thus $cu'(c)$ is bounded. This then means that, in particular, since $d_i \leq \sum_{i=1}^n d_i$ for any particular i, that $d_i' u' \left(\sum_{i=1}^n d_i'\right)$ is bounded by some number, say B. Then it follows that

$$g_i(d) \leq \beta \int B dF(d', d) = \beta B < \infty$$

Thus $g_i(d)$ is bounded.
Standard dynamic programming arguments now apply: define the operator T as

$$T f_i(d) = g_i(d) + \beta \int f_i(d') dF(d', d)$$

From the fact that $g_i(d)$ is both continuous and bounded, and from our assumption that the conditional expectation with respect to F preserves continuity, it is immediate to see that T is a self-map on the space of continuous and bounded functions endowed with the sup norm, with $X = \mathbb{R}_+^n$ (the space of dividends). It is also straightforward to see that T satisfies monotonicity and discounting, for $\beta \in (0, 1)$. Thus it is a contraction mapping in a complete metric space, hence it has a unique fixed point $f_i(d)$. Furthermore, this fixed point is continuous and bounded by construction. $\square$

An economic rationale for the iterative procedure that leads to a unique pricing function can be provided: think of an economy where agents have heterogeneous beliefs or perceptions of how dividends evolve, and what is their distribution. They will naturally price the asset in different ways. However, as time passes and these agents observe the successive realisations of dividends, they eventually learn the true distribution of dividends and a unique price perception function arises. Also note that nothing else regarding the pricing function can be said, without further restrictions on the primitives.

1.2.6 Examples

Example 1 - Linear Utility

Assume that $u(c) = c$. In this case, $u'(c) = 1$, so that $\phi_i(d) = f_i(d)$. The pricing function is then easily characterised. From the Euler Equation:

$$\phi_i(d) = \beta \mathbb{E}_{d'}(d_i'|d) + \beta \mathbb{E}_{d'}[\phi_i(d')|d]$$

Iterating forward and replacing leaves us with

$$\phi_i(d) = \sum_{s=1}^{n} \beta^s \mathbb{E}_{d_i^{(s)}}[d_i^{(s)}|d] + \beta^n \mathbb{E}_{d^{(n)}}[\phi_i(d^{(n)})|d]$$

where the Law of Iterated Expectations is repeatedly applied. Taking the limit as $n \to \infty$ we obtain

$$\phi_i(d) = \sum_{s=1}^{\infty} \beta^s \mathbb{E}_{d_i^{(s)}}[d_i^{(s)}|d] + \lim_{n\to\infty} \beta^n \mathbb{E}_{d^{(n)}}[\phi_i(d^{(n)})|d]$$

Now, the last term is commonly disposed by assuming a *no bubbles condition* or making use of a *Transversality Condition*. We will see this in more detail later on. For now, it is sufficient to note that if ϕ is bounded, then its expectation is bounded, hence as $\lim_{n\to\infty} \beta^n = 0$, the limit term is equal to zero. Thus we are left with the familiar asset pricing equation: price equals the discounted present value of dividends

$$\phi_i(d) = \sum_{s=1}^{\infty} \beta^s \mathbb{E}_{d_i^{(s)}}[d_i^{(s)}|d]$$

Example 2 - Single Asset, iid Returns

If there is a single asset in this economy, there is a single dividend as well, hence $c = d$. This allows us to write the Euler Equation as

$$u'(d)\phi(d) = \beta \int u'(d')[\phi(d') + d']dF(d', d)$$

If, furthermore, returns are iid, then the conditional expectation becomes an unconditional one, as the distribution of dividends next period should not depend on the value of dividends this period. That is, $F(d', d) = F(d')$. Thus

$$u'(d)\phi(d) = \beta \int u'(d')[\phi(d') + d']dF(d')$$

Notice then that the RHS does not depend on d (the state) in any way - it is a constant, Δ.

$$u'(d)\phi(d) = \Delta$$

This should also hold for all periods (as the problem is stationary). Hence we have that

$$\begin{aligned}
\Delta &= \beta \int u'(d')\phi(d')dF(d') + \beta \int u'(d')d'dF(d') \\
&= \beta \int \Delta dF(d') + \beta \int u'(d')d'dF(d') \\
\Rightarrow \Delta &= \frac{\beta \int u'(d')d'dF(d')}{1-\beta}
\end{aligned}$$

and then

$$\phi(d) = \frac{1}{1-\beta} \int \beta \frac{u'(d')}{u'(d)} d'dF(d')$$

Note that the price is equal to the present value of an infinite stream of dividends, discounted at a stochastic discount factor. This expected value is constant as returns are iid! If, for example, we assumed log utility, then we would simply get

$$\phi(d) = \frac{\beta}{1-\beta} d$$

1.2.7 Properties of the Price Function

Let us stick, for now, to the one asset case. We are usually interested in exploring certain properties of asset prices, such as how they change when dividends, risk aversion or volatility change. Their reaction to changes in dividends can be computed as an elasticity. In the iid case, the price is equal to a constant divided by $u'(d)$, hence this price-to-dividend elasticity is simply

$$\frac{\phi'(d)d}{\phi(d)} = -\frac{u''(d)d}{u'(d)}$$

equal to the **Coefficient of Relative Risk Aversion**. Thus the result is straightforward: the more risk averse people are, the more the price of the asset increases *vis-a-vis* an increase in dividends. This may sound counter-intuitive: after all, dividends are iid, so high dividends today do not predict high dividends tomorrow, so why are people buying the asset? The reason is linked to the lack of alternative saving instruments in this economy: if people are risk-averse, then an increase in dividends does not cause a large increase in consumption, as people would prefer to save. However, the only way to save is by purchasing the asset. Thus, faced with a temporary increase in income, agents flock to save and purchase the asset, raising its price.

Let us consider now the general case in which returns are not necessarily iid. From the fact that

$$\phi(d) = \frac{f(d)}{u'(d)}$$

we easily obtain that

$$\frac{\phi'(d)d}{\phi(d)} = \frac{f'(d)d}{f(d)} - \frac{u''(d)d}{u'(d)}$$

The first term was absent in the iid case because $f(d)$ was a constant, hence $f'(d) = 0$. It is known as the *information effect*, and embodies the information that current dividends have regarding future returns for the asset. If current dividends are very good predictors of future returns, this term will tend to have a relatively larger importance. Naturally, in the iid case current returns carry no information regarding future returns, hence this term was zero. The second term is the *risk-aversion effect*, with the same interpretation as before. Note that the above expression raises a fundamental question: in the general case, what sign should we expect for the price to dividend elasticity? In general, we have no idea on what should the sign of $f'(d)$ be (even though we naturally know that $d, f(d) > 0$), and the result crucially hinges on the sign of this term. Fortunately, we can get some structure by imposing further assumptions.

Lemma 1.2.2. *Let F (the transition function) be differentiable and satisfying $0 < -F_2 < F_1$. Consider any function $\psi(x)$ with $0 \le \psi'(x) \le \psi'_M$ for all x (i.e., its derivative is bounded above). Then*

$$0 \le \frac{d}{dy}\int \psi(y')dF(y', y) \le \psi'_M$$

$0 < -F_2 < F_1$ means that dividends are positively autocorrelated: if dividends are high today, $F_2 < 0$ means that the probability of getting higher dividends tomorrow increases (as, for the same value of today's dividends, the mass to the left decreases). The idea is to retrieve the sign of $f'(d)$ from our original functional equation

$$f(d) = g(d) + \beta \int f(d')dF(d', d)$$

by imposing restrictions on the function $g(d)$. Recall that

$$g(d) = \beta \int du'(d)dF(d', d)$$

If we can impose bounds on $g'(d)$, we may be able to bound $f'(d)$ as well (golden rule of dynamic programming: the fixed point inherits most of the properties of the instantaneous payoff function, so it's not surprising that when utility is concave the value function tends to be concave as well, etc.). Our aim is to use the above lemma on $du'(d)$. Note that the derivative of this term is

$$\frac{\partial}{\partial x}[xu'(x)] = u'(x) + xu''(x) = u'(x)\left[1 + \frac{xu''(x)}{u'(x)}\right] = u'(x)[1 - R(x)]$$

where $R(x)$ is the coefficient of relative risk aversion. If we manage to show that the above term is bounded, then we can proceed.

Assumption 1.2.4. *Marginal utility is bounded, $b \leq u'(x) \leq a$ for (b, a) constant.*

From concavity, we have that $R(x) \geq 0$. Thus the above assumptions implies that $[1-R(x)]u'(x)$ is bounded as well. This is not a scandalous assumption: we had concluded before that, under bounded utility, $cu'(c) \leq B \in \mathbb{R}$. Thus $u'(c) \leq \frac{B}{c} \in \mathbb{R}$, as consumption is usually positive with concave utility. Without loss of generality, assume that marginal utility is always positive, hence $b = 0$. Now, there are three interesting cases that must be analysed separately: $R(x) < 1, R(x) = 1$ and $R(x) > 1$. Here, we shall focus mainly on $R(x) < 1$ which, note, is the case in which individuals are less risk averse and, hence, utility tends to be less concave. The lemma then implies that if $[1 - R(x)]u'(x) \in [0, a]$ (which is ensured by our assumption on marginal utility and $R(x) \leq 1$) we have that

$$g'(d) = \beta \int [1 - R(d')]u'(d')dF(d', d) \Rightarrow g'(d) \in [0, \beta a]$$

That is, $g'(d)$ is bounded. This allows us, then, to determine bounds on $f'(d)$, as

$$\frac{d}{dy}Tf(y) = g'(y) + \frac{d}{dy}\beta \int f(y')dF(y', y)$$

(where I changed dividend notation to y to avoid confusion). Noting that we seek to impose bounds on the derivative of the *fixed point*, we can take advantage of the fact that T is a contraction mapping and adopt an iterative procedure: start with some guess

$f_0(y)$. Let $f_k(y) = T^k f_0(y)$. Standard properties of contraction mappings tell us that $\lim_{k\to\infty} T^k f_0(y) = f(y)$. Therefore, start with $f_0 \in CB(X)$ and note that

$$f_1(y) = Tf_0(y) = g(y) + \beta \int f_0(y')dF(y', y)$$

and

$$f_2(y) = Tf_1(y) = g(y) + \beta \int [g(y') + \beta \int f_0(y'')dF(y'', y')]dF(y', y)$$

Proceeding with the iterations

$$f_k(y) = g(y) + \sum_{s=1}^{k-1} \beta^s \int g(y')dF^{(s)}(y', y) + \beta^k \int f_0(y')dF^{(k)}(y', y)$$

where we define

$$F^{(k+1)}(y', y) = \int F(y', z)dF^{(k)}(z, y)$$

That is, the distribution $k + 1$ periods ahead. We know that, by taking the limit as $k \to \infty$, the LHS becomes the fixed point (what we are interested in). Does the second term go away? Yes, as f_0 was assumed to be a bounded function! Thus our fixed point f satisfies

$$f(y) = g(y) + \sum_{s=1}^{\infty} \beta^s \int g(y')dF^{(s)}(y', y)$$

Taking derivatives with respect to y

$$f'(y) = g'(y) + \sum_{s=1}^{\infty} \beta^s \frac{d}{dy} \int g(y')dF^{(s)}(y', y)$$

Now, recall that, from our lemma (assuming, of course, that $0 < -F_2 < F_1$, as all other assumptions are satisfied by the restrictions imposed on the primitives) since $g'(y)$ is bounded we have that

$$\frac{d}{dy} \int g(y')dF(y', y) \leq \beta a$$

Similarly,

$$\frac{d}{dy} \int g(y')dF^{(2)}(y', y) = \frac{d}{dy} \int \int g(y'')dF(y'', y')dF(y', y) \leq \int \beta a dF(y', y) = \beta a$$

and, more generally, for any s

$$\frac{d}{dy} \int g(y')dF^{(s)}(y', y) \leq \beta a$$

Thus

$$f'(y) \leq \sum_{s=0}^{\infty} \beta^s \beta a = \frac{\beta a}{1 - \beta}$$

Furthermore, the lemma can also be applied "from below", as $g'(y) \geq 0$, the same logic applies to establish that

$$f'(y) \geq 0$$

Thus we have bounded $f'(y) \in [0, \beta a/(1-\beta)]$, for the case in which $R(x) < 1$. This accomplishes our objective as, then

$$\frac{\phi'(d)d}{\phi(d)} = -\frac{u''(d)d}{u'(d)} \geq 0$$

Telling us that the price of the asset will be procyclical. Should we expect this? Yes: if there is positive autocorrelation in dividends, higher dividends today imply a high likelihood of high dividends tomorrow. Therefore, as consumers are not 'too much risk averse', they flock to buy the asset and its price increases.
What if $R(x) > 1$, so that consumers are more risk-averse? In this case, it is possible to show that for some lower bound b we will have

$$\frac{\beta b}{1-\beta} \leq f'(y) \leq 0$$

Hence the result regarding the sign of the elasticity becomes ambiguous: agents are extreme consumption smoothers, and they know that if dividends are high today, they will tend to be high in the future. Therefore, they sell excess asset holdings, driving the price down. On the other hand, they are risk-averse and want to insure consumption, thus there is also a motive to purchase the asset (as it is the only vehicle for savings in this economy). Finally, if $R(x) = 1$, it is easily seen that $f'(y) = 0$ and the information effect disappears. This is the case, for example, with log utility, in which income and substitution effects cancel each other.

1.3 Mehra and Prescott 1985 - The Equity Premium: a Puzzle

Between 1889 and 1978, the average return on equity was of about 7%, whereas the average return on riskless bonds was of 1%. Thus there was an average *equity premium* of 6%, in terms of returns. This difference in returns is rather hard to explain with standard, Arrow-Debreu economies: if there is such a large difference in returns, why have people been holding bonds at all? The natural explanation for this phenomenon involves risk: equity is much risky than bonds, hence the premium. However, standard Walrasian models with standard utility and market structure assumptions require implausible high degrees of risk aversion in order to replicate a premium of this magnitude. The level of risk aversion that is compatible with the equity premium is several orders of magnitude greater than the degree of risk aversion that is usually found in micro data, from experiments, etc.

Mehra and Prescott investigate the causes of this puzzle using a very simply model, inspired by the Lucas Tree model but with some key differences.

1.3.1 Environment and Model Set-up

As said, the model is very similar to the Lucas 1978 model. A key difference, however, is that while Lucas worked in a stationary environment, the authors sought to replicate US economy behaviour for a century. Therefore, they needed to incorporate growth in the model, modeled as growth of endowments, or dividends of the stock.

Endowments/dividends grow at a stochastic rate each period. Let d_t denote the dividend at time t, and x_t its growth rate. Then

$$d_{t+1} = x_{t+1} d_t$$

where $x_{t+1} \in \{\gamma_1, \dots, \gamma_n\}$ takes values in a finite state space. The growth rate of the dividend is modeled as a Markov Chain P where

$$\mathbb{P}(x_{t+1} = \gamma_j | x_t = \gamma_i) = p_{ij}$$

Preferences are time and state separable as in the Lucas model. Instantaneous utility is assumed to be CRRA

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

with $\sigma \in (0, \infty)$ and $\beta \in (0, 1)$ as the discount factor.

In terms of **market and asset structure**, there is a single risky asset, a Lucas tree, that pays a dividend as described above. There is also a one period risk-free bond, that pays one unit of consumption with certainty. It is assumed that initial dividends are positive, $d_0 > 0$, and that gross growth rates are always positive as well, $\gamma_i > 0, \forall i$. This implies that dividends will always be non-negative. d_t is observed at the beginning of each period, before any trades are undertaken. It is also assumed that the matrix $M = [m_{ij}]$, where

$$m_{ij} = \beta p_{ij} \gamma_j^{1-\sigma}$$

is a stable matrix, in the sense that all of its eigenvalues are inside the unit circle. This implies that

$$\lim_{k \to \infty} M^k = 0$$

Finally, $P = [p_{ij}]$ is assumed to be ergodic, so that $p_{ij} > 0, \forall i, j$. Ergodicity of P ensures that any random variable defined as a deterministic mapping of x_t (the state variable) has a well defined theoretical mean. Not only this, but the time average of that variable converges in probability to its expected value. That is, for any $y_t = x_t \bar{y}$ we will observe

$$\frac{1}{T} \sum_{t=0}^{T} y_t \to_p \mathbb{E}(y_t)$$

where $\mathbb{E}(y_t)$ is well-defined.

1.3.2 The Individual Problem

As in Lucas, Mehra and Prescott adopt a representative agent formulation. The sequence problem can be described as

$$\max_{\{c_t, a_{t+1}^s, a_{t+1}^b\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{SP}$$

subject to

$$\begin{aligned} c_t + \phi_t^s a_{t+1}^s + \phi_t^b a_{t+1}^b = (\phi_t^s + d_t) a_t^s + a_t^b \\ c_t \geq 0 \\ a_{t+1}^i \geq 0, i = s, b \end{aligned}$$

where all variables with s superscript refer to the stock and with the b superscript refer to the bond. Expectations are taken with respect to the sequence of dividend growth rates, $\{x_t\}_{t=1}^{\infty}$. Assuming an interior solution, the (necessary) First Order Conditions with respect to stock and bond holdings are

$$\begin{aligned} a_{t+1}^s &: u'(c_t)\phi_t^s = \beta \mathbb{E}_t[u'(c_{t+1})(\phi_{t+1}^s + d_{t+1})] \\ a_{t+1}^b &: u'(c_t)\phi_t^b = \beta \mathbb{E}_t[u'(c_{t+1})] \end{aligned}$$

Note that the two conditions are essentially the same: in the case of the bond, there is no dividend and the resell value is normalised to one unit of consumption in prices at $t+1$.

1.3.3 Equilibrium and Prices

In a representative agent framework with monotonic preferences, the goods market clears trivially by feasibility

$$c_t = d_t, \forall t \geq 0$$

This condition can be replaced in the FOC's, so as to transform them into functional equations that may allow us to extract pricing functions, as in the Lucas setting. In particular, for the stock we get

$$\phi_t^s = \beta \mathbb{E}_t \frac{u'(d_{t+1})}{u'(d_t)} d_{t+1} + \beta \mathbb{E}_t \frac{u'(d_{t+1})}{u'(d_t)} \phi_{t+1}^s$$

Replacing recursively leaves us with

$$\phi_t^s = \mathbb{E}_t \sum_{j=1}^{n} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j} + \beta^n \frac{u'(d_{t+n})}{u'(d_t)} \phi_{t+n}$$

Taking limits as $n \to \infty$ and assuming no bubbles (so that the price is bounded and the last term vanishes), we obtain a typical present-discounted value of dividends asset pricing equation

$$\phi_t^s = \mathbb{E}_t \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j}$$

Mehra and Prescott seek to find a Recursive Competitive Equilibrium, where the problem can be described as a stationary one. In a RCE, as we have seen with Lucas, prices are a function of the state variables. However, while in the stationary Lucas environment, the only state variable was the dividend, now we must also account for the growth rate (as its current value helps us in predicting the future). Therefore, they postulate the existence of a function Φ^s such that

$$\phi_t^s = \Phi^s(x_t, d_t) = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j} | x_t, d_t \right]$$

Given Mehra and Prescott's assumptions on utility, the stochastic discount factor can be handily computed as

$$\frac{u'(d_{t+j})}{u'(d_t)} = \left(\frac{d_t}{d_{t+j}} \right)^{\sigma}$$

Why is this useful? Because it makes the pricing function homogeneous of degree one in dividends. Noting that

$$d_{t+1} = x_{t+1} d_t$$

we have that

$$d_{t+j} = \prod_{i=1}^{j} x_{t+i} d_t$$

That is,

$$
\begin{aligned}
\Phi^s(x_t, d_t) &= \mathbb{E}_t\left[\sum_{j=1}^{\infty} \beta^j \left(\frac{d_t}{d_{t+j}}\right)^{\sigma} d_{t+j}|x_t, d_t\right] \\
&= \mathbb{E}_t\left[\sum_{j=1}^{\infty} \beta^j \left(\frac{d_t}{\prod_{i=1}^{j} x_{t+i} d_t}\right)^{\sigma} \prod_{i=1}^{j} x_{t+i} d_t|x_t, d_t\right] \\
&= d_t \mathbb{E}_t\left[\sum_{j=1}^{\infty} \beta^j \left(\frac{1}{\prod_{i=1}^{j} x_{t+i}}\right)^{\sigma} \prod_{i=1}^{j} x_{t+i}|x_t, d_t\right] \\
&= d_t \mathbb{E}_t\left[\sum_{j=1}^{\infty} \beta^j \left(\prod_{i=1}^{j} x_{t+i}\right)^{1-\sigma} |x_t\right]
\end{aligned}
$$

Note that the growth rate is independent of the level of dividends (it only depends on its own past value, given that it is modeled as a Markov chain). This means that the utility specification allows us to express the price of the asset as a constant (which is state specific!) times the dividend.

This motivates the following definition: let $\phi^s(d, i) = \Phi^s(x_t, d_t)$ when $d_t = d$ and $x_t = \gamma_i$. Note that everything we need to price the asset is summarised on the current state and the level of the dividend.

This given, the Euler Equations for prices can then be written, with recursive notation, as

$$
\begin{aligned}
d^{-\sigma} \phi^s(d, i) &= \beta \sum_{j=1}^{n} p_{ij} (\gamma_j d)^{-\sigma} [\phi^s(\gamma_j d, j) + \gamma_j d] \\
d^{-\sigma} \phi^b(d, i) &= \beta \sum_{j=1}^{n} p_{ij} (\gamma_j d)^{-\sigma}
\end{aligned}
$$

where, note, we take advantage of the Markovian structure of the endowment process, which allows us to explicitly compute the expectation. $\gamma_j d$, given today's endowment d is tomorrow's dividend when the economy is on state j. Canceling the $d^{-\sigma}$ term and rearranging leaves us with neater price functions

$$
\begin{aligned}
\phi^s(d, i) &= \beta \sum_{j=1}^{n} p_{ij} \gamma_j^{-\sigma} [\phi^s(\gamma_j d, j) + \gamma_j d] \\
\phi^b(d, i) &= \beta \sum_{j=1}^{n} p_{ij} \gamma_j^{-\sigma}
\end{aligned}
$$

Note that the price of a risk free bond is **independent** of the dividend level d. It depends only on the current state (through the transition probabilities). Thus there will be as many bond prices as states: n in our case. There are, then, n possible values (at most) for the risk-free rate, which is defined as

$$R_f(i) = \frac{1}{\phi^b(i)}$$

A similar 'trick' can be performed with the stock price. Recall that we have established it as being homogeneous of degree one in the dividend level. This means that $\phi^s(d,i) = d\phi_i^s$ for some constant $\phi_i^s, \forall i$. Thus we can rewrite the stock pricing equation as

$$\phi_i^s d = \beta \sum_{j=1}^{n} p_{ij} \gamma_j^{-\sigma} [\phi_j^s \gamma_j d + \gamma_j d]$$

Allowing us to eliminate the dividend level to obtain

$$\phi_i^s = \beta \sum_{j=1}^{n} p_{ij} \gamma_j^{1-\sigma} (\phi_j^s + 1)$$

Thus we have n constants ϕ_i^s, each of them described by an equation which depends on those n constants. Therefore, we have just obtained n equations for n unknowns, totally describing the stock prices $\phi^s = (\phi_1^s, \ldots, \phi_n^s)$. So, in some sense, the reasoning is parallel to that of the bond prices: even though stock prices are not independent of the level of dividends, they will be equal to a constant times the level of the dividend, and that constant depends only on the current state. Thus, in some sense, there are also n possible prices for the risky asset.

We now proceed to solve this system. We can write equation i of the system as

$$\phi_i^s = \beta \sum_{j=1}^{n} p_{ij} \gamma_j^{1-\sigma} \phi_j^s + \beta \sum_{j=1}^{n} p_{ij} \gamma_j^{1-\sigma}$$

Stacking over i leaves us with a system in matrix form

$$\phi^s = M\phi^s + a$$

where ϕ^s is $n \times 1$, M is $n \times n$ and a is $n \times 1$, described as

$$M = [m_{ij}] = [\beta p_{ij} \gamma_j^{1-\sigma}]$$
$$a = [a_i] = [\beta \sum_{j=1}^{n} p_{ij} \gamma_j^{1-\sigma}]$$

Now, if $I - M$ is nonsingular, we can solve the system as

$$\phi^s = (I - M)^{-1} a$$

Do we have invertibility of that matrix? Note that

$$(I - M)^{-1} = \sum_{j=0}^{\infty} M^j$$

and, by assumption, $\lim_{k\to\infty} M^k = 0$. Any geometric sum satisfying this converges [3]. Thus the inverse exists and is finite, allowing us to solve the system for the vector of price constants.

1.3.4 Asset Returns

Let r^s_{ij} denote the return on equity from state i to state j. It is constructed as

$$r^s_{ij} = \frac{\phi^s(\gamma_j d, j) + \gamma_j d}{\phi^s(d, i)} = \frac{\gamma_j(1+\phi^s_j)}{\phi^s_i}$$

where the second to third equality take advantage of homogeneity of the price on d. Thus, as we could somehow expect, equity returns do not depend on the level of dividends.

The **conditional return on equity at state** i can then be computed as

$$R^s_i = \mathbb{E}(r^s_{ij}|x_t = \gamma_i) = \sum_{j=1}^{n} p_{ij} r^s_{ij} = \frac{1}{\phi^s_i}\sum_{j=1}^{n} p_{ij}\gamma_j(1+\phi^s_j)$$

As we have seen, the return on bonds depends only on its current price which, in turn, depends only on the current state. Thus we can automatically obtain the conditional return from the current price:

$$R^b_i = \frac{1}{\phi^b(d,i)} = \frac{1}{\beta\sum_{j=1}^{n} p_{ij}\gamma_j^{-\sigma}}$$

Unconditional returns can be simply obtained by averaging conditional returns at each state using, for example, the stationary distribution of P. The transition matrix is ensured to have a unique stationary distribution given our ergodicity assumption of $p_{ij} > 0, \forall i, j$ (Ljungqvist & Sargent, p. 33). Given the stationary distribution π satisfying $\pi' = \pi' P$, unconditional returns can be obtained as

$$R^s = \sum_{i=1}^{n} \pi_i R^s_i$$
$$R^b = \sum_{i=1}^{n} \pi_i R^b_i$$

The equity premium can then be computed as $E = R^s - R^b$.

[3] Obviously, this is not true in general. This is a specific feature of geometric sums, and emanates from the fact that convergence in a geometric sum is equivalent to the common ratio being strictly within the unit circle, $r \in (-1, 1)$. But, this is equivalent to $\lim_{n\to\infty} r^n = 0$.

1.3.5 Model Calibration and the Equity Premium Puzzle

Calibration involves assigning values to the deep parameters of the model. In the Mehra and Prescott environment, calibration involves assigning values to the following elements:

1. **Preferences** - The discount factor β and the coefficient of relative risk aversion σ.
2. **Technology** - The values that can be taken by the growth rate of the dividend, $\{\gamma_i\}_{i=1}^n$ (note that this involves choosing a n as well), and the associated transition matrix P.

In their article, M&P assume two states: a low growth state (or recession) and a high growth state (or boom), $\gamma \in \{\gamma_L, \gamma_H\}$. They specify $\gamma_H = \gamma_1 = \bar{\gamma} + \delta$ and $\gamma_L = \gamma_2 = \bar{\gamma} - \delta$. The transition matrix is assumed to be symmetric

$$P = \begin{bmatrix} \mu & 1-\mu \\ 1-\mu & \mu \end{bmatrix}$$

Calibration of the parameters describing the endowment process, $(\bar{\gamma}, \delta, \mu)$ can now be achieved by taking the average growth rate of consumption, the standard deviation of consumption and the first order autocorrelation of the consumption process, respectively. Calibration can be undertaken through the method of moments:

$$\begin{aligned}
\mathbb{E}[\mathbb{E}_t \gamma_{t+1}] &= \sum_{i=1}^n \pi_i \sum_{j=1}^n p_{ij} \gamma_j = \bar{\gamma} \\
\mathbb{V}(\gamma) = \mathbb{E}[\mathbb{E}_t(\gamma_{t+1} - \bar{\gamma})^2] &= \sum_{i=1}^n \pi_i \sum_{j=1}^n p_{ij} (\gamma_j - \bar{\gamma})^2 = \delta^2 \\
\mathbb{C}(\gamma_t, \gamma_{t+1}) &= (2\mu - 1)\delta^2
\end{aligned}$$

Using US data, M&P assign

$$\begin{aligned}
\bar{\gamma} &= 1.018 \\
\delta &= 0.036 \\
2\mu - 1 &= -0.14 \Rightarrow \mu = 0.43
\end{aligned}$$

using yearly data on consumption growth.

How to calibrate the preference parameters (β, σ)? In principle, one should pick values for those parameters that would yield results that are consistent with the data (now that we have explicit formulas for the returns on stocks and bonds). However, it turns out that this is not an easy task: for standard values of the discount factor, the following table summarises the results generated by different choices of σ, the coefficient of relative risk aversion

Clearly, no *acceptable* levels of relative risk aversion yield the equity premium that is observed in the data. Furthermore, and noting that E is increasing in σ, but is also

Table 1.1: Returns and Equity Premia for Different levels of risk aversion

σ	R^s	R^b	$R^s - R^b$
1	2.83	2.70	0.13
2	4.58	4.30	0.28
3	6.27	5.79	0.48
4	7.89	7.18	0.71
5	9.42	8.45	0.97
6	10.88	9.62	1.26
7	12.24	10.67	1.57
8	13.52	11.60	1.92
9	14.70	12.41	2.29
10	15.79	13.10	2.69

associated with increasing average returns on R^s and R^b, attaining an equity premium of 6% would require unconditional returns on both stocks and bonds that are considerably larger than those observed in the data.

The equity premium puzzle, the inability of standard models to replicate this return differential is associated with another closely related inconsistency (between the model and the data): the **risk-free rate puzzle**. For agents to be willing to hold a significant amount of bonds, in the model, they demand a rate of return that is much higher than 1%, so how come people are holding bonds in the first place?

The main problem is associated with the fact that the parameter σ plays, in fact, a **double role**: on the one hand, it represents the coefficient of relative risk aversion, how much consumption is the individual willing to trade across states of the world. Therefore, as σ increases, the individual prefers to move away from stocks and prefers to buy bonds. In equilibrium, this means that the price of stocks will decrease and $R^s \uparrow$. However, σ is also the inverse of the **Elasticity of Intertemporal Substitution**: how much consumption is the individual willing to trade across periods. Therefore, as $\sigma \uparrow$, the EIS decreases and the individual becomes less willing to trade consumption across periods. Thus, to induce more saving, the price of the bond must go down and $R^f \uparrow$. This is the main cause of the 'puzzle'.

It is then clear that, to solve this issue, one should be separating these two motives: risk aversion (consumption across states of the world) and intertemporal substitution (consumption across periods). This is achievable, for example, by using Epstein-Zin Preferences, which are able to parametrise risk aversion and the EIS independently. The main problem with these preferences is that, analytically, they can prove to be extremely cumbersome to work with, making several models (even simple ones) intractable.

1.4 Other Approaches to the Equity Premium Puzzle

Ever since Mehra and Prescott's article, the equity premium puzzle (EPP) has been thoroughly studied by economists. In particular, a large effort has been devoted to attempts at reconciling the EPP with standard, simple asset pricing and macroeconomic models.

1.4.1 Kocherlakota, 1996 - The Equity Premium: It's still a Puzzle

A simple test for Mehra and Prescott's claims regarding the existence of several puzzles can be undertaken by imposing very little structure on the model, and using readily available data. Recall that, for a generic asset pricing model, the Euler Equations describing equity and bond prices could be written as

$$u'(c_t)\phi_t^s = \beta \mathbb{E}_t[u'(c_{t+1})(\phi_{t+1}^s + d_{t+1})]$$
$$u'(c_t)\phi_t^b = \beta \mathbb{E}_t[u'(c_{t+1})]$$

Now, let R_{t+1}^i denote the return on asset $i = \{s, b\}$, as defined in the previous section (gains at $t+1$ divided by price at t), and define the stochastic discount factor (relative to period t) as

$$M_{t+1} := \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

The Euler Equations can then be rewritten in terms of returns and the stochastic discount factor

$$1 = \mathbb{E}_t \beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{\phi_{t+1}^s + d_{t+1}}{\phi_t^s} = \mathbb{E}_t M_{t+1} R_{t+1}^s$$
$$1 = \mathbb{E}_t \beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{1}{\phi_t^b} = \mathbb{E}_t M_{t+1} R_{t+1}^b$$

Subtracting the two, one from the other, gives us an indifference, or non-arbitrage, condition for the two assets

$$\mathbb{E}_t M_{t+1}(R_{t+1}^s - R_{t+1}^b) = 0$$

In the margin, an optimising investor should be indifferent between putting his or her own money in the stock or in the bond, and the returns should be utility-equivalent, once adjusted to risk by the stochastic discount factor. Furthermore, and regardless of the composition of his own portfolio, the investor should be indifferent between saving and/or eating an additional unit of consumption

$$\mathbb{E}_t M_{t+1} R_{t+1}^b - 1 = 0$$

These two (conditional) moment restrictions can be tested using the data that is available on bond and stock returns. Constructing the stochastic discount factor does, however, require specification of an utility function (if we assume CRRA preferences, for example, everything that is needed is data on consumption growth). In general, and for very agnostic utility specifications, the data tends to reject the above two restrictions. Kocherlakota (*The Equity Premium: It's still a puzzle*, 1996, JEL) rejects both restrictions using US data. In particular, he finds that

$$\begin{aligned} \mathbb{E}_t M_{t+1}(R^s_{t+1} - R^b_{t+1}) &> 0 \\ \mathbb{E}_t M_{t+1} R^b_{t+1} - 1 &< 0 \end{aligned}$$

Given what we know, the first inequality should be expected: it is nothing more than the equity premium puzzle, once again. Even when adjusted for risk and the stochastic discount factor, the return on equity tends to significantly exceed that of bonds. Regarding the second inequality, it is indicative of the risk-free puzzle: it indicates that individuals are *saving too much*: the marginal benefit of savings exceeds its cost, and individuals are over-investing in bonds.

1.4.2 Abel, 1988 - Stock Prices and Dividend Risk

This article changes slightly the Mehra and Prescott set-up by assuming that the growth rate of dividends, instead of following a Markov Chain, is iid and lognormal. The author also abstracts from specifying a process for endowments, and specifies a direct exogenous process for consumption growth (this will not make much of a difference in a representative agent framework, as we should expect). Starting from the standard Euler Equations in an environment with the same market structure (perishable good, one risky stock and one risk-free bond)

$$\begin{aligned} u'(c_t)\phi^s_t &= \beta \mathbb{E}_t[u'(c_{t+1})(\phi^s_t + d_{t+1})] \\ u'(c_t)\phi^b_t &= \beta \mathbb{E}_t[u'(c_{t+1})] \end{aligned}$$

The author makes the following assumptions

1. Utility is CRRA, $u'(c_t) = c_t^{-\sigma}$.
2. Consumption growth follows an exogenous process, $c_{t+1} = e^{g^c_{t+1}} c_t$, where g^c_{t+1} is a random variable.
3. Dividends also follow a random growth process, $d_{t+1} = e^{g^d_{t+1}} d_t$, where g^d_{t+1} is a random variable.
4. (g^c_{t+1}, g^d_{t+1}) is a iid vector (over time) and jointly normal, with parameters $(\mu_c, \mu_d, \Sigma_c, \Sigma_d, \rho)$.

Let $\gamma^i_{t+1} := e^{g^i_{t+1}}$ for $i = c, d$ denote the gross rate of growth. Given these assumptions, we can write the Euler Equation for the stock as

$$\phi^s_t = \beta \mathbb{E}_t[(\gamma^c_{t+1})^{-\sigma}(\phi^s_{t+1} + d_{t+1})]$$

and, just like in Mehra and Prescott's model, stock prices will be linear on the current level of dividends. To see this, assume that $\phi_t^s = \omega d_t$ for some constant ω. From the Euler Equation, we can easily confirm that

$$\begin{aligned}\omega d_t &= \beta \mathbb{E}_t[(\gamma_{t+1}^c)^{-\sigma} d_{t+1}(1+\omega)] \\ \Leftrightarrow \omega &= \beta \mathbb{E}_t[(\gamma_{t+1}^c)^{-\sigma}(1+\omega)\gamma_{t+1}^d] \\ \Rightarrow \omega &= \frac{\beta \mathbb{E}_t[(\gamma_{t+1}^c)^{-\sigma}\gamma_{t+1}^d]}{1-\beta \mathbb{E}_t[(\gamma_{t+1}^c)^{-\sigma}\gamma_{t+1}^d]}\end{aligned}$$

Given that $(\gamma_{t+1}^c, \gamma_{t+1}^d)$ is a deterministic mapping of two jointly distributed random variables that are independent across time, we can simply remove the t subscript from the above expectations and treat them as constants. Thus ω is indeed a constant, and prices are indeed linear on the level of dividends. Furthermore, while in Mehra and Prescott, we were happy at this point (to determine the constants that multiply the level of prices), we can now go even further, by using the assumptions that were imposed on the process for growth rates. This allows us to more deeply investigate the underlying motives of the equity premium.
So far, we have not touched the subject of goods market equilibrium. Clearly, in equilibrium, we must observe $c_t = d_t$. This implies that $\gamma_{t+1}^c = \gamma_{t+1}^d$ for all t and, consequently, these two random variables are the same: $\mu_c = \mu_d, \Sigma_c = \Sigma_d, \rho = 1$. Letting γ_{t+1} denote the growth rate in this economy, we can write

$$\omega = \frac{\beta \mathbb{E}[\gamma_{t+1}^{1-\sigma}]}{1-\beta \mathbb{E}[\gamma_{t+1}^{1-\sigma}]}$$

And, using the properties of the lognormal distribution, from the fact that $\log \gamma \sim \mathcal{N}(\mu, \Sigma^2)$ we have that

$$\mathbb{E}(\gamma^{1-\sigma}) = \exp\left[(1-\sigma)\mu + \frac{1}{2}(1-\sigma)^2\Sigma^2\right]$$

This allows us to derive a closed form for the price of the stock:

$$\phi_t^s = \frac{\beta \exp\left[(1-\sigma)\mu + \frac{1}{2}(1-\sigma)^2\Sigma^2\right]}{1-\beta \exp\left[(1-\sigma)\mu + \frac{1}{2}(1-\sigma)^2\Sigma^2\right]} d_t$$

The bond price is also easily derived:

$$\phi_t^b = \beta \mathbb{E}\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right] = \beta \mathbb{E}(\gamma^{-\sigma}) = \beta \exp\left[-\sigma\mu + \frac{1}{2}\sigma^2\Sigma^2\right]$$

and note that it is constant over time, in opposition to the Mehra and Prescott case, in which it depended on the current state. This is a consequence of time-independence of dividends: given that the current state does not help predict the future state (while in the Markovian environment, it did).

Some remarks are then in order:

1. If $\sigma = 1$ (log utility), then we get

$$\phi_t^s = \frac{\beta}{1-\beta} d_t$$

Exactly the same result we obtained in the special case of iid dividends in Lucas '78.

2. How does the stock price react to changes in the average growth rate, μ? Note that

$$\frac{\partial \phi_t^s}{\partial \mu} = \phi_t^s \frac{1-\sigma}{1 - \beta \exp\left[(1-\sigma)\mu + \frac{1}{2}(1-\sigma)^2 \Sigma^2\right]}$$

Given that the term in the denominator is positive (as otherwise, the price of the asset would be negative, thus this is in fact a regularity condition that must be satisfied by the parameters), the sign of the term will depend on whether $\sigma > 1$ or $\sigma < 1$. Thus, if agents are risk-averse, $\sigma > 1$, the price of the asset will tend to decrease. Why? Recall that σ doubled as the EIS: thus agents with a high σ have a low elasticity of intertemporal substitution and prefer rather stable consumption. Therefore, if the growth rate of the economy increases, they do not need as much of the asset in order to stabilise consumption, thus they sell the asset. Besides, selling the asset makes them less exposed to risk. If agents do not care about consumption smoothing, $\sigma < 1$, then they flock to purchase the asset as dividends are expected to grow faster. If $\sigma = 1$, there is no effect: under log-utility, the income and substitution effects cancel with each other. Therefore, the two effects that were described (the first is the substitution, whereas the second is the income one) exactly offset each other.

3. What is the impact of greater growth variance, $\Sigma \uparrow$? Once again, note that

$$\frac{\partial \phi_t^s}{\partial \Sigma} = \phi_t^s \frac{(1-\sigma)^2 \Sigma}{1 - \beta \exp\left[(1-\sigma)\mu + \frac{1}{2}(1-\sigma)^2 \Sigma^2\right]} \geq 0$$

The term is strictly positive unless $\sigma = 1$. This is counter-intuitive, especially if $\sigma > 1$. We would expect risk-averse agents to flee from the risky stock as the variance of its dividend rises, and the price to go down. However, recall that this is also the variance of the consumption process. Therefore, agents feel the need to insure more as the variance of consumption increases. Insurance is only obtained through the risk-free bond and the risky asset. However, the price of the risk-free bond will rise much more, hence some agents are left to purchase the risky asset in order to save (this can be checked analytically).

4. What is the direct impact of risk aversion σ on prices? It can be checked that

$$\text{sgn}\left(\frac{\partial \phi_t^s}{\partial \sigma}\right) = -\text{sgn}[\mu - (\sigma - 1)\Sigma^2]$$

This, naturally, will depend on the value of σ and on how the average growth rate compares to its own volatility. Note, however, that for a very high Σ, regardless of σ and μ the derivative will tend to be positive, highlighting the precautionary motive for saving.

In terms of returns, we also obtain closed forms:

$$R^b_{t+1} = \frac{1}{\phi^b} = \frac{1}{\beta \exp\left[-\sigma\mu + \frac{1}{2}\sigma^2\Sigma^2\right]}$$
$$R^s_{t+1} = \frac{\phi^s_{t+1} + d_{t+1}}{\phi^s_t} = \frac{(1+\omega)d_{t+1}}{\omega d_t} = \left(1 + \frac{1}{\omega}\right)\gamma_{t+1}$$

As we would expect, the return on bonds is constant (as its price is constant). Further note that the return on equity depends only on the growth rate of dividends, not on their level. Therefore, the conditional return will be equal to the unconditional return (i.e. will not depend on t):

$$\begin{aligned}\mathbb{E}_t R^s_{t+1} &= \frac{\mathbb{E}_t \gamma_{t+1}}{\beta \exp\left[(1-\sigma)\mu + \frac{1}{2}(1-\sigma)^2\Sigma^2\right]} \\ &= \frac{\exp\left[\mu + \frac{1}{2}\Sigma^2\right]}{\beta \exp\left[(1-\sigma)\mu + \frac{1}{2}(1-\sigma)^2\Sigma^2\right]} \\ &= \frac{1}{\beta}\exp\left[\sigma\mu + \frac{\sigma}{2}(2-\sigma)\Sigma^2\right] = \mathbb{E}R^s_{t+1}\end{aligned}$$

We are now ready to look at the implications generated by this model for the equity premium. Note that instead of looking at the difference, we can rather analyse the ratio of the returns between stocks and bonds. Or, rather, as it is conveniently suggested by our functional forms, at the log of the ratio:

$$\log \mathbb{E}R^s_{t+1} = -\log\beta + \sigma\mu + \sigma\Sigma^2 - \frac{1}{2}\sigma^2\Sigma^2$$
$$\log R^b_{t+1} = -\log\beta + \sigma\mu - \frac{1}{2}\sigma^2\Sigma^2$$

The expression for the log return on the bond is very transparent in the sense that it explicitly decomposes all of the effects that are at play in determining prices/returns: first comes impatience, the log of the discount factor: as it increases, people save more and the return on the bond goes down (as its price increases). Then, the average growth rate of dividends has a positive impact on the return: as dividends grow faster, the first-order effect is the income one, people move away from the bond and towards the stock. To compensate for that, the price of the bond comes down and its return increases. Finally, we have the second-order, or substitution effect: as volatility rises, people prefer the bond to the stock, thus its price increases and returns decrease.

The equity premium can be computed as the difference of the logs, which is simply:

$$\log(EP) = \sigma\Sigma^2$$

The extra term in the expression for the log of equity returns. Once again, we extract the root of the premium: since equity is risky, it must pay an excess return over bonds. In the case of this model, this is linear in risk-aversion!

In terms of calibration, note that we have data for everything except for σ. Thus we stand, roughly, on the same grounds as Mehra and Prescott. The author sets $R^b_{t+1} \simeq 1$, thus its log is approximately zero. The expected return on equity, as in Mehra and Prescott, is approximately 7%. The standard deviation of the growth rate of consumption is set as $\Sigma^2 = 0.0036^2$. This allows us to solve for σ in

$$\log \mathbb{E} R^s_{t+1} - \log R^b_{t+1} = \sigma \Sigma^2$$

generating $\sigma = 52$ as a requirement for obtaining an equity premium of 7%. Naturally, calibration suspicions fall on the value taken by Σ^2, which appears to be too small in our case. In order to obtain an acceptable value for the coefficient of relative risk aversion, say $\sigma = 2$, we would need consumption to be 26 times more volatile in the data than it actually is!

1.4.3 Hansen-Jagannathan, 1991 - Constructing empirical bounds

Hansen and Jagannathan (*Implications of Security Market Data for Models of Dynamic Economies*, 1991, JPE) go even further, by imposing even less structure on the model. Instead of assuming a particular functional form for utility, they rather work with a very general discount factor in a generic asset pricing setting, and try to recover from the data reasonable properties that a SDF should satisfy to be consistent with real data. The assumptions imposed on the SDF correspond to very loose assumptions on utility: concave, time and state separable, etc.

1. Sharpe Ratio based Bounds

The asset pricing Euler Equations can, as we have seen, be written as

$$\mathbb{E}_t M_{t+1} R^i_{t+1} = 1$$

for $i = s, b$. In particular, given that the bond is risk-free, its return at $t+1$ depends only on its current price, hence is known at t. This allows us to remove the bond return from the conditional expectation and write it as

$$\mathbb{E}_t M_{t+1} = \frac{1}{R^b_{t+1}}$$

For the equity equation, we can use the covariance formula to expand the expectation into

$$\mathbb{E}_t R^s_{t+1} \mathbb{E}_t M_{t+1} + \mathbb{C}_t(M_{t+1}, R^s_{t+1}) = 1$$

So that, using the fact that the expected value of the SDF equals the known inverse return of the bond, we can rearrange to obtain an expression for excess returns:

$$\mathbb{E}_t R^s_{t+1} - R^b_{t+1} = -\frac{\mathbb{C}_t(M_{t+1}, R^s_{t+1})}{\mathbb{E}_t M_{t+1}}$$

This simple (and extremely agnostic) derivation is extremely enlightening: why is there an equity premium to begin with? Clearly, we can only expect stocks to yield greater returns than bonds if the covariance term is negative. What does this mean? If there is negative covariance between the SDF and the stock return, this means that the stock tends to yield high returns when the SDF is low. But the SDF is low when future marginal utility is high with respect to the current period - that is, when future consumption is high. Thus the stock is an extremely **bad form of insurance**: it yields high returns when consumption is high (and, therefore, high returns are not needed). The stock is a procyclical stock, does not allow hedging against bad periods, hence agents demand a premium to hold it!

The covariance formula tells us that

$$\mathbb{E}_t R^s_{t+1} - R^b_{t+1} = -\frac{\sigma_t(M_{t+1})\sigma_t(R^s_{t+1})\rho_t(M_{t+1}, R^s_{t+1})}{\mathbb{E}_t M_{t+1}}$$

where σ_t are conditional standard deviations and ρ_t is the conditional coefficient of correlation. Rearranging the expression, dividing both sides by the standard deviation of equity returns, leaves us with

$$\frac{\mathbb{E}_t R^s_{t+1} - R^b_{t+1}}{\sigma_t(R^s_{t+1})} = -\rho_t(M_{t+1}, R^s_{t+1})\frac{\sigma_t(M_{t+1})}{\mathbb{E}_t M_{t+1}}$$

The LHS of the above expression, the excess return of the stock with respect to the risk-free rate divided by its standard deviation is known as the **Sharpe Ratio** - the higher the better, as it means that the security pays a higher return at less risk. The term on the RHS, that multiplies the coefficient of correlation, can be seen as the **market price of risk**: the standard deviation of the SDF divided by its expected value, which is the same as the risk free rate multiplied by the average risk in the market! Given that the coefficient of correlation must lie between zero and one, we can use the above expression to impose a bound on what values are acceptable for the market price of risk:

$$\left|\frac{\mathbb{E}_t R^s_{t+1} - R^b_{t+1}}{\sigma_t(R^s_{t+1})}\right| \leq \left|\frac{\sigma_t(M_{t+1})}{\mathbb{E}_t M_{t+1}}\right|$$

The above condition provides us with a quick and easy to run test on whether a SDF derives from reasonable assumptions or not: the LHS, the Sharpe Ratio, can be easily computed from the data. Thus a reasonable utility specification (generating a reasonable SDF) should satisfy the above condition. If the model is not consistent with it, then the model is not likely to ever be consistent with the equity premium that is observed in the data. Considering that some securities can have extremely high Sharpe Ratios, a SDF should be volatile enough to account for this. It should be noted that the utility specification and calibration adopted by Mehra and Prescott violated the above condition.

2. Alternative Specifications for the Bounds

Consider a more general framework with n securities. The asset pricing equations for each security impose that $\mathbb{E}_t M_{t+1} x_{i,t+1} = 1$, where $x_{i,t+1}$ denotes the return on the i-th security. This equation, as we know, should be valid for any SDF and time and state separable utility specification. Now, consider the projection of the SDF on the returns of the n securities, that is:

$$M = a + b_1 x_1 + \ldots + b_n x_n + \varepsilon$$

or

$$M = a + b'x + \varepsilon$$

This projection problem can be solved as

$$\min_{a,b} \mathbb{E}[(M - a - b'x)^2]$$

From convexity of the above function, the following FOC are both necessary and sufficient

$$\mathbb{E}(M - a - b'x) = 0$$
$$\mathbb{E}[x(m - a - b'x)] = 0$$

Note that the first FOC is imposing that $\mathbb{E}(\varepsilon) = 0$, whereas the second is $\mathbb{C}(\varepsilon, x) = 0$ (as the expected value of ε is 0 by the first one). This is a simple linear projection exercise, whose solution is given by

$$b = \mathbb{C}(x,x)^{-1}\mathbb{C}(x,M) = \mathbb{C}(x,x)^{-1}[\mathbb{E}(Mx) - \mathbb{E}(M)\mathbb{E}(x)]$$

From the asset pricing equation, we know that $\mathbb{E}_t(M_{t+1}x_{i,t+1}) = 1$ for any i, t. Therefore, by the Law of Iterated Expectations, we obtain that $\mathbb{E}(Mx_i) = 1 \Rightarrow \mathbb{E}(Mx) = \iota$, where $x = (x_1, \ldots, x_n)$. Thus our linear projection coefficient b can be written as

$$b = \mathbb{C}(x,x)^{-1}\mathbb{C}(x,M) = \mathbb{C}(x,x)^{-1}[\iota - \mathbb{E}(M)\mathbb{E}(x)]$$

Furthermore, by construction of our model

$$M = a + b'x + \varepsilon \Rightarrow \mathbb{V}(M) = \mathbb{V}(b'x) + \mathbb{V}(\varepsilon) + \mathbb{C}(x'b, \varepsilon)$$

where the covariance term is equal to zero, by construction of the linear projection coefficient and linear projection error term. Thus

$$\mathbb{V}(M) \geq \mathbb{V}(b'x) = b'\mathbb{C}(x,x)^{-1}b$$

Replacing for our expression for b, leaves us then with

$$\mathbb{V}(M) \geq [\iota - \mathbb{E}(M)\mathbb{E}(x)]'\mathbb{C}(x,x)^{-1}[\iota - \mathbb{E}(M)\mathbb{E}(x)]$$

which is known as the **Hansen-Jagannathan Bound** on the variance of the stochastic discount factor. Once again, we observe that in order to be consistent with the empirical evidence on the equity premium, the SDF should be volatile unknown, as its variance is bounded below. For any generic M, one can use data on returns of several securities to construct the above bound, which must be satisfied for any SDF in a reasonable model. The conditions gives us a lower envelope of combinations of means and standard deviations that are considered acceptable by the data.

Further bounds can be constructed, based on excess returns. Take any two assets i, j and construct the excess returns between the two, $z_{ij} := x_i - x_j$. The asset pricing equation implies the following moment condition

$$\mathbb{E}(M z_{ij}) = 0$$

for any pair of assets. The whole exercise can then be repeated to obtain

$$b = -\mathbb{C}(z, z)^{-1}\mathbb{E}(M)\mathbb{E}(z)$$

So that the H-J bound for excess returns becomes

$$\sigma_M = \sqrt{\mathbb{V}(M)} \geq [\mathbb{E}(z)'\mathbb{C}(z, z)^{-1}\mathbb{E}(z)]^{1/2}\mathbb{E}(M)$$

Note that while the previous bound on returns will tend to be a parabola (or, at least, have quadratic features), the restriction on excess returns will be a straight line. Thus this bound is, in some sense, less stringent than the previous one. Naturally, an acceptable SDF should belong on the intersection of the two bounded areas.

Chapter 2

Money in Classical Economies and Transversality Conditions

2.1 Introduction

This section addresses the introduction of an asset without any value in an Arrow-Debreu economy with complete markets. As we expect, in the absence of fundamental value, this asset will not be traded at all. As we know, the value of fiat money, in OLG economies for example, is based on a bubble. The demographic structure of these economies is such that money acquires value by becoming a savings vehicle when no other instruments are available: once a Lucas tree/stock is introduced in such economies, the value of money disappears - such assets 'burst' the money bubble. Therefore, if the structure of the economy is such that no bubbles are possible, money can never be valued.

The discussion on money and asymptotic asset valuation also provides a good motivation to discuss some further properties of classical dynamic problems, in particular when are Transversality Conditions necessary and sufficient.

The environment is as follows:

1. **Time** is discrete and the horizon is infinite.

2. The **demographics** consist of a finite number N of infinitely lived agents.

3. In the **commodity space**, there is a single consumption good in each period. There is no production nor storage: this is a pure exchange economy. There are no firms nor government either.

4. Each household owns a deterministic stream of **endowments** of the consumption good, denoted by $e^i = \{e^i_t\}_{t=0}^{\infty}$.

5. Households have **preferences** over infinite sequences of consumption, $c^i = \{c_t^i\}_{t=0}^{\infty}$. These preferences are time and state separable

$$U(c^i) = \sum_{t=0}^{\infty} \beta^t u^i(c_t^i)$$

where the discount factor is $\beta \in (0, 1)$, and the instantaneous utility function u^i is strictly increasing, concave and continuously differentiable. Furthermore, we impose a Inada condition

$$\lim_{c \to 0} u'^i(c) = \infty$$

6. In terms of the **information structure**, there is no uncertainty in the model.

7. The **market structure** follows a standard Arrow-Debreu formulation, with time 0 trading.

2.2 Money in a Classical Arrow-Debreu Setting

Definition 2.2.1. *An **allocation** is a sequence (of sequences) $\{c^i\}_{i=1}^{N}$, specifying a consumption profile for each agent.*

Letting $\{p_t\}_{t=0}^{\infty}$ denote a sequence of time 0 Arrow-Debreu prices, the agent's problem can be written as

$$\max_{c^i} U^i(c^i) = \max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) \tag{HP}$$

subject to the Arrow-Debreu budget constraint and non-negativity conditions

$$\sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t e_t^i$$
$$c_t^i \geq 0, \forall t \geq 0$$

Definition 2.2.2. *An **Arrow-Debreu Competitive Equilibrium** (ADE) is an allocation $\{c^i\}_{i=1}^{N}$ and a sequence of prices $\{p_t\}_{t=0}^{\infty}$ such that*

1. *Given $\{p_t\}_{t=0}^{\infty}$, c^i solves (HP) for each $i = \{1, \ldots, N\}$.*

2. *Markets clear at every period*

$$\sum_{i=1}^{N} c_t^i = \sum_{i=1}^{N} e_t^i, \forall t \geq 0$$

It should be noted that the standard existence arguments apply here: for an ADE to exist, (HP) must have a well-defined solution for each agent. This requires the time 0 value of the endowment to be finite for each agent, that is

$$\sum_{t=0}^{\infty} p_t e_t^i < \infty, \forall i$$

This requirement is crucial for the following lemma:

Lemma 2.2.1. *In an ADE, the value of the aggregate endowment* $\sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} e_t^i$ *is finite.*

Proof. As remarked, a necessary condition for existence of ADE is that the endowment is finite for each agent. Furthermore, note that $p_t e_t^i \geq 0, \forall t, i$ by construction of ADE: preferences are strictly monotonic on the sole consumption good, hence all prices must be strictly positive. The endowments are nonnegative by assumption. Naturally, the value of the aggregate endowment at $t = 0$ is obtained by summing the value of each agent's endowment across agents. Given that we are summing N convergent series composed of non-negative elements, rearrangements should not matter as per Dirichlet's Rearrangement Theorem. This means that we can write:

$$\sum_{i=1}^{N} \sum_{t=0}^{\infty} p_t e_t^i = \sum_{t=0}^{\infty} \sum_{i=1}^{N} p_t e_t^i = \sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} e_t^i$$

given that the first LHS is finite, as it is a finite sum of finite elements, so is the last RHS, the value of the aggregate endowment. □

2.2.1 The Value of Money

The above discussion provides us with the arsenal to prove the main result:

Proposition 2.2.1. *Fiat money has no value in any ADE.*

Proof. By contradiction, suppose not and suppose that there is a ADE with valued fiat money. Let q_t denote the (spot) price of money at period t. We can then write the Arrow-Debreu budget constraint for each agent as

$$\sum_{t=0}^{\infty} p_t (c_t^i + q_t m_{t+1}^i) \leq \sum_{t=0}^{\infty} p_t (e_t^i + q_t m_t^i)$$

The Lagrangian for an individual consumer's problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u^i(c_t^i) + \lambda^i \sum_{t=0}^{\infty} p_t (e_t^i + q_t m_t^i - c_t^i - q_t m_{t+1}^i)$$

The FOC being:

$$
\begin{gathered}
c_t^i : u'^i(c_t^i) = \lambda^i p_t \\
m_{t+1}^i : p_t q_t \lambda^i = \lambda^i p_{t+1} q_{t+1}
\end{gathered}
$$

We ignore non-negativity conditions on money holdings because, by market clearing, if money is valued, then at least one agent must hold m_{t+1}^i. Without loss of generality, assume that it is the agent whose problem we are looking at. Clearly, the FOC for money implies that

$$p_t q_t = p_{t+1} q_{t+1} = \rho, \forall t$$

Thus, aggregating budget constraints, we obtain that

$$\sum_{i=1}^{N} \sum_{t=0}^{\infty} p_t (c_t^i + q_t m_{t+1}^i) = \sum_{i=1}^{N} \sum_{t=0}^{\infty} p_t (e_t^i + q_t m_t^i)$$

Note that, as we have seen, if we ensure that all terms of a summation are non-negative, we are free to change the order of the summation. I claim that all terms on the RHS are non-negative. From monotonicity of preferences, we trivially have that $p_t > 0, \forall t \geq 0$. Thus if we show that $q_t \geq 0$, we are done. Assume that $q_t < 0$ for some t. Then, by monotonicity on c_t^i, any agent will choose to hold an infinite amount of money, and this cannot be an equilibrium. Thus $q_t \geq 0$. This allows us then to swap the summations on the RHS to obtain

$$\sum_{i=1}^{N} \sum_{t=0}^{\infty} p_t (c_t^i + q_t m_{t+1}^i) = \sum_{t=0}^{\infty} \left[p_t \sum_{i=1}^{N} e_t^i + p_t q_t \sum_{i=1}^{N} m_t^i \right]$$

As we have seen, in any Arrow-Debreu equilibrium, the value of the aggregate endowment is finite. Therefore, $\sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} e_t^i < \infty$. From $p_t q_t = \rho, \forall t \geq 0$, we obtain, however, that

$$\sum_{t=0}^{\infty} p_t q_t \sum_{i=1}^{N} m_t^i = \sum_{t=0}^{\infty} \rho \sum_{i=1}^{N} m_t^i = \infty$$

Given that $m_t^i > 0$ for at least one agent. Therefore, we have that

$$\sum_{i=1}^{N} \sum_{t=0}^{\infty} p_t (c_t^i + q_t m_{t+1}^i) = \infty$$

or, given that there is a finite number of agents, the LHS of some agent's budget constraint is infinite. This cannot happen in equilibrium, as it allows for $c_t^i = \infty$ for some i. Therefore, given a fixed amount of money in the economy, we must have $q_t = 0$ for at least one t. Given that $p_t q_t = p_{t+1} q_{t+1}$ and $p_t > 0, \forall t$, this then implies that $q_t = 0, \forall t \geq 0$. Therefore, money is worthless in equilibrium. □

An alternative proof goes as follows:

Proof. Once again, write the Arrow-Debreu time 0 budget constraint with money holdings as

$$\sum_{t=0}^{\infty} p_t(c_t^i + q_t m_{t+1}^i) \leq \sum_{t=0}^{\infty} p_t(e_t^i + q_t m_t^i)$$

Or rearrange it to obtain

$$\sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t e_t^i + \sum_{t=0}^{\infty} p_t q_t (m_{t+1}^i - m_t^i)$$

The claim is proved if we show that the second term on the RHS is equal to zero. If the term is equal to zero, then we establish that money does not increment the amount of resources that can be consumed by the agent, hence has zero (utility) value. Let that term be denoted by m^i. Assume then, by contradiction, that m^i. In an ADE, physical balance must be satisfied, that is

$$\sum_{i=1}^{N} c_t^i = \sum_{i=1}^{N} e_t^i, \forall t \geq 0$$

Since market clearing holds for any period, by Walras' Law (as preferences are monotonic)

$$p_t \sum_{i=1}^{N} c_t^i = p_t \sum_{i=1}^{N} e_t^i$$

Summing over t

$$\sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} c_t^i = \sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} e_t^i$$

Now, what happens if we sum the individual budget constraints across agents?

$$\sum_{i=1}^{N} \sum_{t=0}^{\infty} p_t c_t^i = \sum_{i=1}^{N} \sum_{t=0}^{\infty} p_t e_t^i + \sum_{i=1}^{N} m^i$$

Since we can rearrange any of the above infinite series, having $m^i > 0$ for any i will immediately contradict either market clearing/feasibility in every period or the aggregate budget constraint. For the two to be consistent, we must have $m^i = 0, \forall i$. □

Both of the above proofs crucially rely on finiteness of the value of the aggregate endowment at an ADE to show that money cannot have any value. In fact, it is finiteness of the value of the aggregate endowment that rules out bubbles in this economy, and hence strips money from any value it may acquire. This highlights the main reason why money tends to have value in OLG models: due to the double infinity (of agents and periods), the value of the endowment may not be finite. This gives room for bubbles to arise, and hence for money to acquire value.

2.3 Pricing Money in the Lucas Model

It could be argued that the previous set-up, a pure exchange Arrow-Debreu economy, is too restrictive for money to have an effect and, henceforth, be valued. We now proceed to show that this result is rather robust: even in less restrictive economies, where a wider variety of assets may cohabit, money is still likely to become a worthless asset given its lack of fundamental value. To see this, consider the original Lucas Tree economy, with two relevant adjustments:

1. There is a single tree paying a deterministic stream of dividends, $\{d_t\}_{t=0}^{\infty}$.

2. There is a government that issues fiat money to pay for government acquisitions of fruit. Consider a sequence of government expenditures $\{g_t\}_{t=0}^{\infty}$. At $t = 0$, the government places M dollars (per capita) in the economy to acquire fruit. Let ϕ_t^m denote the value of a dollar in terms of fruit at time t (how many units of fruit are needed to buy a dollar). Clearly, this will be the inverse of the price level, $p_t = \frac{1}{\phi_t^m}$. Government expenditures are

$$\begin{aligned} g_0 &= M\phi_0^m \\ g_t &= 0, \forall t \geq 1 \end{aligned}$$

Assume that $\{d_t\}$ is uniformly bounded away from zero. That is, $\exists \bar{d} : 0 < \bar{d} \leq d_t, \forall t \geq 0$.

The agent will now choose a sequence of consumption, and may choose either to save through shares of the tree or money. Therefore, there will be a nontrivial portfolio allocation problem in this economy. Once again, we work with a representative agent formulation. The problem, in a sequential budget constraint formulation, can be stated as

$$\max_{\{c_t, s_{t+1}, m_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{HPL}$$

subject to

$$\begin{aligned} c_t + \phi_t^s s_{t+1} + \phi_t^m m_{t+1} &= (\phi_t^s + d_t) s_t + \phi_t^m m_t \\ c_t, s_{t+1}, m_{t+1} &\geq 0 \\ s_0 = 1; m_0 &= 1 \end{aligned}$$

Note that the agent starts with no money holdings: they are injected by the government in the economy at $t = 0$. Our definition of equilibrium will be the following:

Definition 2.3.1. *For a given sequence $\{d_t, g_t\}_{t=0}^{\infty}$, a **Competitive Equilibrium** is an allocation $\{c_t, s_{t+1}, m_{t+1}\}_{t=0}^{\infty}$ and a price system $\{\phi_t^s, \phi_t^m\}_{t=0}^{\infty}$ such that*

1. *Given prices and the sequence $\{d_t, g_t\}_{t=0}^{\infty}$, the allocation solves (HPL).*

2. Markets clear at every period

$$\begin{aligned} Goods &: c_t + g_t = d_t \\ Stocks &: s_{t+1} = 1 \\ Money &: m_{t+1} = M \end{aligned}$$

Definition 2.3.2. *A Competitive Equilibrium which, furthermore, satisfies $\phi_t^m > 0$ for all $t \geq 0$ is called a* ***Monetary Equilibrium*** *or a* ***Competitive Equilibrium with Valued Fiat Money****.*

Note that, from the way we have specified the stream of government expenditures, we should observe, from feasibility and goods market clearing:

$$\begin{aligned} c_0 &= d_0 - g_0 \\ c_t &= d_t, \forall t \geq 1 \end{aligned}$$

As hinted in the previous section, the introduction of a Lucas Tree will kill off any bubbles and make money worthless. We now proceed to show this in several steps. We begin with the following lemma that tells us that if the value of money ever hits zero, then it will remain zero forever.

Lemma 2.3.1. *If $\phi_t^m = 0$ for some t, then $\phi_{t+s}^m = 0, \forall s \geq 1$.*

This is, in some sense, a converse of the classical result on bubbles that says that a bubble can only exist today if there is some future period/state in which a bubble will exist. In this case, we are saying that if there is no bubble today, there will be no bubble in the future.

Proof. Suppose not, and suppose that $\phi_{t+s}^m > 0$ for some $s > 0$ and $\phi_t^m = 0$. Then, this cannot be an equilibrium, as the agent will set $m_{t+1} = \infty$ and the budget constraint will be unbounded at $t+s$ (the agent will be able to choose infinite consumption). Therefore, we must have $\phi_t^m > 0$, a contradiction. □

The next lemma is closer to the classical result on bubbles:

Lemma 2.3.2. *If $\phi_t^m = 0$ for some t, then $\phi_{t-s}^m = 0, \forall 0 \leq s \leq t$.*

So, if the money has zero value today, it cannot have had any value in the past. If there is no bubble today, there cannot have existed any bubble in the past.

Proof. The logic is similar: consider $\phi_t^m = 0$ and $\phi_{t-1}^m > 0$. Clearly, the agent will not hold any money at $t-1$. For the money market to clear, the price must therefore decrease. But the agent is not willing to hold any amount of money for any positive price. Therefore, the price must be equal to zero for the agent to hold the money that is in the economy. By induction, we conclude that this must hold to any period prior to t. □

We have therefore established that if the price of money ever hits zero, at any point in time, then it is **always** equal to zero (that is, before and after that period). Thus, to show that money has no value, it is enough to show that money has no value *for a single arbitrary period.*

We can write the Lagrangian for the consumer, explicitly incorporating all the constraints (such as the non-negativity ones) as

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + \lambda_t [(\phi_t^s + d_t)s_t + \phi_t^m m_t - c_t - \phi_t^s s_{t+1} - \phi_t^m m_{t+1}] + \mu_t m_{t+1} + \sigma_t s_{t+1} + \gamma_t c_t \right\}$$

The FOC are the following:

$$\begin{aligned} c_t &: u'(c_t) = \lambda_t - \gamma_t \\ s_{t+1} &: \lambda_t \phi_t^s = \beta \lambda_{t+1}(\phi_{t+1}^s + d_{t+1}) - \sigma_t \\ m_{t+1} &: \lambda_t \phi_t^m = \beta \lambda_{t+1} \phi_{t+1}^m - \mu_t \end{aligned}$$

Plus complementary slackness conditions

$$\begin{aligned} \lambda_t [(\phi_t^s + d_t)s_t + \phi_t^m m_t - c_t - \phi_t^s s_{t+1} - \phi_t^m m_{t+1}] &= 0 \\ \mu_t m_{t+1} &= 0 \\ \sigma_t s_{t+1} &= 0 \\ \gamma_t c_t &= 0 \end{aligned}$$

The non-negativity multipliers $(\mu_t, \sigma_t, \gamma_t)$ are equal to zero whenever the corresponding control variable is strictly positive. Given that we have assumed an Inada condition, and the dividend is uniformly bounded away from zero, we can safely assume that $\gamma_t = 0$. Foreseeing what will happen in equilibrium, we can also assume that $\sigma_t = 0$, and will be primarily concerned with the μ_t term. Combining the FOC's for c_t, s_{t+1} we obtain the typical Euler Equation

$$u'(c_t)\phi_t^s = \beta u'(c_{t+1})(\phi_{t+1}^s + d_{t+1})$$

Note that since this is a fully deterministic problem, there are no expectations hanging around. From the FOC for money, we can also obtain an expression for the multiplier on the non-negativity constraint on money holdings:

$$\mu_t = \phi_t^m u'(c_t) - \beta \phi_{t+1}^m u'(c_{t+1})$$

Replacing in the complementary slackness condition leaves us with

$$[\phi_t^m u'(c_t) - \beta \phi_{t+1}^m u'(c_{t+1})] m_{t+1} = 0$$

or, from the fact that $u' > 0, \forall c$

$$\left[\phi_t^m - \beta \frac{u'(c_{t+1})}{u'(c_t)} \phi_{t+1}^m \right] m_{t+1} = 0$$

We have put together the Euler Equation for money and the complementary slackness condition, thus allowing us to describe the optimal money holdings with a single equation. Note also that the rearrangement allowed us to write this expression as a function of the stochastic discount factor. From the Euler Equation for stocks, we know that we can write it as...

$$\beta\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\phi_t^s}{\phi_{t+1}^s + d_{t+1}} = \frac{1}{R_{t+1}^s}$$

...the inverse of the return on the stock. Thus our condition for money becomes

$$\left[\phi_t^m - \frac{1}{R_{t+1}^s}\phi_{t+1}^m\right] m_{t+1} = 0$$

Now, if this economy has a Monetary Equilibrium, then $m_{t+1} > 0$ for some agent. Thus the term in brackets must equal zero. That is

$$R_{t+1}^s = \frac{\phi_{t+1}^m}{\phi_t^m}$$

The 'return on money' must equal the return on the stock. Recall that we characterised the price of money as the inverse of the price level, $\phi_t^m = \frac{1}{p_t}$. Thus the above statement is equivalent to

$$R_{t+1}^s = \frac{p_t}{p_{t+1}} = \frac{1}{\pi_{t+1}}$$

The return on the stock should be equal to the inverse of the gross rate of inflation in a monetary equilibrium! This is the non-arbitrage condition that ensures indifference between the two assets, and it is quite intuitive: the stock pays a real return - the dividend. For the agent to be indifferent between the stock and money, with the latter paying no real return, it must be that money is gaining value in terms of capital gains at the same rate as the stock pays the dividend. Thus, if the (net) return on the stock is positive, the price of money must be going up: one unit of money today must be worth less than one unit of money tomorrow! In other words, the economy must be experiencing deflation.

Now, take the Euler Equation for equity and iterate it forward to obtain an expression for the stock price:

$$\phi_t^s = \lim_{n\to\infty}\sum_{j=1}^{n-1}\beta^j\frac{u'(c_{t+j})}{u'(c_t)}d_{t+j} + \lim_{n\to\infty}\beta^n\phi_{t+n}^s\frac{u'(c_{t+n})}{u'(c_t)}$$

Let us assume that the bubble term is equal to zero, that is, prices are bounded in equilibrium. This allows us then to write the price of the stock as

$$\phi_t^s = \sum_{j=1}^{\infty}\beta^j\frac{u'(c_{t+j})}{u'(c_t)}d_{t+j}$$

Now, recall from feasibility that $c_0 = d_0 - g_0$ and $c_t = d_t, \forall t \geq 1$. This means that

$$\phi_t^s = \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j}, \forall t \geq 1$$
$$\phi_0^s = \sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_0 - g_0)} d_{t+j} = \beta \frac{u'(d_1)}{u'(d_0 - g_0)} (\phi_s^1 + d_1)$$

where, recall, $g_0 = \phi_0^m M$, which highlights the fact that the monetary and equity sides of the model are not completely independent: the price of equity at period 0 will be influenced by the price of money at $t = 0$ (but not afterwards).

So, in order to solve for equity prices, we need to pin down money prices. From the complementary slackness condition, we had concluded that

$$\phi_{t+1}^m = R_{t+1}^s \phi_t^m$$

or, more generally, for any $j \geq t$

$$\phi_{t+j}^m = \left(\prod_{i=1}^{j} R_{t+i}^s \right) \phi_t^m$$

This holding for <u>all</u> $t \geq 0$. This allows us to express the price of money at any point in time as the function of the price of money at $t = 0$

$$\phi_t^m = \left(\prod_{i=0}^{t-1} R_{i+1}^s \right) \phi_0^m$$

From the fact that the return on equity equals the (inverse of the) stochastic discount factor, we get that

$$\begin{aligned}
\phi_t^m &= \left(\prod_{i=0}^{t-1} \frac{u'(c_i)}{\beta u'(c_{i+1})} \right) \phi_0^m \\
&= \frac{u'(c_0)}{\beta u'(c_1)} \frac{u'(c_1)}{\beta u'(c_2)} \cdots \frac{u'(c_{t-1})}{\beta u'(c_t)} \phi_0^m \\
&= \frac{u'(c_0)}{\beta^t u'(c_t)} \phi_0^m \\
&= \frac{u'(d_0 - g_0)}{\beta^t u'(d_t)} \phi_0^m
\end{aligned}$$

and note that the price of money in period 0 is also 'hidden' in $g_0 = M\phi_0^m$. So if we determine ϕ_0^m, we can determine the price of money at any point in time, as we have the sequence of dividends.

Note that from $u' > 0, \forall c > 0$, it follows that $\phi_0^m > 0 \Rightarrow \phi_t^m > 0, \forall t \geq 0$. Similarly, if $\phi_0^m = 0$, then $\phi_t^m = 0, \forall t \geq 0$. We take advantage of this fact to prove the following proposition, that tells us that money will be worthless in this economy.

Proposition 2.3.1. *In any equilibrium, $\phi_t^m = 0, \forall t \geq 0$.*

Proof. Note that the sequence $\left\{\frac{u'(d_0-g_0)}{u'(d_t)}\right\}_{t=0}^{\infty}$ is bounded away from zero. Why? Recall that $0 < \bar{d} \leq d_t, \forall t$. Therefore $u'(d_t) \leq u'(\bar{d}) < \infty, \forall t$, and the sequence is always strictly positive. This then means that

$$\frac{u'(d_0 - g_0)}{\beta^t u'(d_t)} \to \infty$$

as $\frac{1}{\beta^t} \to \infty$. This then implies that if $\phi_0^m > 0$, from our previous remark, we will have $\lim_{t\to\infty} \phi_t^m = \infty$. Since the price is unbounded in the limit, if the agent holds $m_{t+1} > 0$, he will be able to attain unbounded consumption, and this cannot be an equilibrium. Thus $\phi_0^m = 0 \Rightarrow \phi_t^m = 0, \forall t \geq 0$ $\square$

We then show that the value of money is equal to zero at any period. Note then that $g_0 = M\phi_0^m = 0$, the government is 'not able to inject' the money in the economy, as no one is willing to buy it at a positive price. Thus we are still in Lucas' original economy: Lucas did not miss anything by not adding money to the model.

The main obstacle to money in this model is the existence of the tree asset: for anyone to be willing to hold money, it must pay the same return of equity, as we have observed. However, this requires the price of money to grow at a fast enough rate so that the capital gains from holding it match the *real* gains from holding the asset (the dividend gains). But then the price of money must explode, and this is not compatible with a well-defined equilibrium. It is relevant to note that while the Lucas tree strengthens the result, money would not survive in its absence either, for the same reason.

2.3.1 Example: Constant Dividends

Suppose that the stream of dividends is constant, $d_t = d, \forall t$. Then, the stochastic discount factor becomes simply:

$$\frac{u'(d_t)}{\beta u'(d_{t+1})} = \frac{1}{\beta}$$

Meaning that the non-arbitrage condition for money is given by

$$\frac{\phi_{t+1}^m}{\phi_t^m} = R_{t+1}^s = \frac{1}{\beta}$$

so that the return on the stock is constant. As we have seen, the growth rate of the price of money is the inverse growth rate of the price level, which in this case will be $\frac{p_t}{p_{t+1}} = \frac{1}{\beta}$. Thus the net inflation rate is $\pi_t = \beta - 1 < 0$ negative, as we had pointed out. The general expression for the price of money at t can be written as

$$\phi_t^m = \beta^{-t}\phi_0^m$$

where we can see that the price of money will be rising *too fast* to match the return on equity. Thus the only possible equilibrium involves setting $\phi_0^m = 0$, so that money is worthless forever. Finally, the price of equity also has a very simple expression:

$$\frac{\phi_{t+1}^s + d}{\phi_t^s} = \frac{1}{\beta} \Rightarrow \phi_t^s \frac{1}{\beta}(1 - \beta L^{-1}) = d$$

which yields

$$\phi_t^s = \frac{\beta d}{1 - \beta}$$

That is, the price of the asset will be constant, which is not surprising in a fully deterministic environment where dividends are constant.

2.4 Transversality Conditions

The discussion on money provides a good environment for discussing Transversality Conditions. Recall that, so far, we have always 'assumed away bubbles' by imposing that prices should be bounded. Therefore, expressions of the kind

$$\lim_{T \to \infty} \beta^T \frac{u'(c_{t+T})}{u'(c_t)} \phi_{t+T}$$

are assumed to be equal to zero - both consumption in the future and prices are bounded, hence the whole expression converges to zero via $\beta \in (0, 1)$.

However, some remarks are in order:

1. It has already been hinted that bubbles may arise in certain circumstances (OLG models with money, for example).
2. Very often, we can actually show that the absence of bubbles is a *consequence* of optimising behaviour by the agent. That is, instead of assuming away bubbles, we can show that this arises due to more fundamental assumptions we impose on the problem.

This section looks more deeply at the conditions under which Euler Equations and Transversality Conditions are necessary or sufficient for optimality.

2.4.1 Solving Difference Equations

Very often in macroeconomics, we encounter difference equations of the type

$$y_t = \beta y_{t+1} + \gamma x_t$$

Assume that $\{x_t\}_{t=0}^{\infty}$ (the exogenous shock sequence) is an element of $[\ell_\infty[0,\infty), d_\infty]$, where $d_\infty = \sup_t |x_t - y_t|$. Also assume that $|\beta| < 1$ and $\gamma \in \mathbb{R}$.

There are several possible approaches to solving the above equation. *Solving* in this context involves isolating the endogenous variable y_t and expressing it as a closed-form function of the exogenous sequence. Two of these common approaches follow:

Contraction Methods

These involve, as the name implies, defining a contraction mapping and taking advantage of the convergence properties to finding a solution. For the above equation, define the operator $T : \ell_\infty \to \ell_\infty$ as

$$[T(y)]_t = \beta y_{t+1} + \gamma x_t$$

That is, given some exogenous sequence $\{x_t\}$ and any arbitrary sequence $\{y_t^0\}$, the mapping T takes $\{y_t^0\}$ and generates a whole new sequence, $\{y_t^1\} := \{[T(y^0)]_t\}$, in the manner that is described above (applying the difference equation to every element of the old sequence).

Claim 2.4.1. *T is a contraction mapping on ℓ_∞.*

Proof. To check this, it is enough to apply Blackwell's Sufficiency Conditions. Note first that ℓ_∞ is the space of bounded sequences. Therefore, T is a self-map on the space of *bounded, natural-valued* functions, which is itself a particular case of a space of bounded functions. This allows us to effectively apply Blackwell, as one of its requirements is that T be a self-map on a space of bounded functions. Note that without further assumptions on β, we can do nothing, as $\beta < 0$ would violate the usual monotonicity assumption. We proceed by studying each case separately:

1. If $\beta \in [0, 1)$, monotonicity follows immediately, as for any $\{y_t\} \geq \{z_t\}$ (meaning that $y_t \geq z_t, \forall t$) we have that

   $$[T(y)]_t = \beta y_{t+1} + \gamma x_t \geq \beta z_{t+1} + \gamma x_t = [T(z)]_t, \forall t$$

   hence $\{[T(y)]_t\} \geq \{[T(z)]_t\}$. In what concerns discounting, note that for any constant $a \geq 0$ and any sequence $\{y_t\} \in \ell_\infty$ we have that

   $$[T(y + a)]_t = \beta(y_{t+1} + a) + \gamma x_t = [T(y)]_t + \beta a$$

   Thus discounting follows from $\beta < 1$. This then establishes T as a contraction mapping.

2. In case $\beta \in (-1, 0)$, we can use an alternative formulation of Blackwell's Sufficient Conditions for a contraction mapping:

   Proposition 2.4.1. *(Alternative Blackwell) Let $X \subseteq \mathbb{R}^n$ and let $B(X)$ be the space of real valued bounded functions defined on X, endowed with the* sup *norm. Let $T : B(X) \to B(X)$ be an operator satisfying* *(a) For any $f, g \in B(X)$, $f \leq g \Rightarrow (Tf) \geq (Tg)$.*

(b) $\exists \beta \in [0,1)$ such that $T(f-a) \leq (Tf) + \beta a, \forall f \in B(X), a \geq 0$.

Then, T is a contraction with modulus β.

For 'reverse monotonicity', take $\{y_t\} \leq \{z_t\}$. Then

$$[T(y)]_t = \beta y_{t+1} + \gamma x_t \geq \beta z_{t+1} + \gamma x_t = [T(z)]_t$$

since $\beta < 0 \Rightarrow \beta y_{t+1} \geq \beta z_{t+1}$. For 'reverse' discounting, take $a \geq 0$ and $\{y_t\}$, then

$$[T(y-a)]_t = \beta(y_{t+1} - a) + \gamma x_t = [T(y)]_t - \beta a = [T(y)]_t + \tilde{\beta} a$$

where $\tilde{\beta} := -\beta$ and belongs in the $[0,1)$ interval. Thus T is a contraction with modulus $\tilde{\beta}$.

□

Now that we have established T as a contraction mapping, let us proceed to find its fixed point. It exists since (ℓ_∞, d_∞) is a complete metric space, hence the Banach Fixed Point Theorem applies - and we get to know that the fixed point is unique! Define, as already suggested, $y^{k+1} := T(y^k)$, the $k+1$-th iteration starting at some arbitrary sequence y^0. For any sequence of such kind:

$$\begin{aligned}
[T(y^0)]_t &= y_t^1 = \beta y_{t+1}^0 + \gamma x_t \\
[T(y^1)]_t &= y_t^2 = \beta y_{t+1}^1 + \gamma x_t \\
\ldots &= \ldots \\
[T^k(y^0)]_t &= y_t^k = \beta y_{t+1}^{k-1} + \gamma x_t = \beta[\beta y_{t+2}^{k-2} + \gamma x_{t+1}] + \gamma x_t \\
&= \beta^k y_{t+k}^0 + \gamma \sum_{j=0}^{k-1} \beta^j x_{t+j}
\end{aligned}$$

Now, taking the limit $k \to \infty$ gives us, by the properties of the contraction mapping, its fixed point. Let $y_t = \lim_{k\to\infty}[T^k(y^0)]_t$. Then:

$$y_t = \gamma \sum_{j=0}^{\infty} \beta^j x_{t+j} + \lim_{k\to\infty} \beta^k y_{t+k}^0$$

But since $\{y_t^0\} \in \ell_\infty$, it is a bounded sequence. Hence $y_t^0 < \infty, \forall t$. This, along with $|\beta| < 1$ ensures that the limit term disappears, allowing us to obtain the solution to our difference equation

$$y_t = \gamma \sum_{j=0}^{\infty} \beta^j x_{t+j}$$

Forward Substitution

Consider a more general setting in which $\{y_t\}$ need not be bounded. In this world, the limit term need no longer vanish as $k \to \infty$, since y^0_{t+k} could diverge to infinity, for example. In fact, by imposing bounded behaviour, we may be ruling out some viable equilibria in the previous recursive formulation of the problem. While keeping with the assumption that $\{x_t\} \in \ell_\infty$, unboundedness may only arise if the crucial assumption of $|\beta| < 1$ is dropped. Without this assumption, the previous method cannot be exactly repeated, as the contraction mapping as we have defined will not even exist. Suppose, then, that $|\beta| > 1$. In this case, the equation must be solved *backwards* instead of forward. That is, rewrite it as

$$y_{t+1} = \frac{1}{\beta} y_t - \frac{\gamma}{\beta} x_t$$

and since $|1/\beta| < 1$, a contraction mapping argument can now be applied to solve the equation as in the first case.

2.4.2 The Canonical Setting

As it is evident, both methods rely heavily on a recursive structure that breaks down if, for example, the assumption that $\{x_t\} \in \ell_\infty$ is dropped. As stated, by forcing this term to live on a bounded world, we may be ruling out and ignoring equilibria that are potentially relevant to the analysis.

We then seek a more general treatment and conditions under which we can, in fact, assume (or obtain) that the limiting terms vanish.

For this, let us set up the **canonical sequential problem**, in which some decision maker seeks to optimise over some infinite sequence $\{x_t\}_{t=0}^{\infty}$ of endogenous state variables

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{SP}$$

subject to

$$x_{t+1} \in \Gamma(x_t), \forall t \geq 0$$
$$x_0 = \bar{x}_0 \text{ given}$$

The 'deep' primitives of the problem are the following:

1. $X \subseteq \mathbb{R}^n$ is the state space.

2. $\Gamma : X \rightrightarrows X$ is the constraint correspondence.

3. $F : X \times X \to \mathbb{R}$ is the (instantaneous) return function.

We are further going to consider the following assumptions (or a subset of them)

1. (E1) X is a convex subset of $\mathbb{R}^n$, and Γ is nonempty, compact-valued and continuous.
2. (E2) F is bounded and continuous, and $\beta \in (0, 1)$.
3. (M1) $F(\cdot, y)$ is strictly increasing for each y.
4. (M2) Γ is monotonic: $x \leq x' \Rightarrow \Gamma(x) \subseteq \Gamma(x')$.
5. (C1) F is strictly concave.
6. (C2) Γ is convex valued: for every $\lambda \in (0, 1)$ and $x, x' \in X$, we have that $y \in \Gamma(x), y' \in \Gamma(x') \Rightarrow \lambda y + (1 - \lambda)y' \in \Gamma(\lambda x + (1 - \lambda)x')$.
7. (D) F is continuously differentiable on the interior of the constrained set $\{(x, y) \in X \times X : y \in \Gamma(x)\}$.
8. (I) Given (E1), (E2), (M1), (M2), V (the value function for the recursive formulation of SP) is strictly increasing.

Based on this framework, we now proceed to discuss the necessity and sufficiency of the Euler Equation and Transversality Conditions for solving the (SP) problem.

2.4.3 Necessity of the Euler Equation

Suppose that the sequence $\{x_t^*\}_{t=0}^{\infty}$ solves (SP). This implies that there cannot be any feasible and profitable one period deviations from this sequence. That is, given x_t^* yesterday and x_{t+2}^* tomorrow, it must be optimal to choose x_{t+1}^* today, and not anything else. The sequence is, in Game Theoretical language, *unimprovable*. Formally, this means that x_{t+1}^* is the solution to the *one period problem*, defined as follows

$$\max_y \{F(x_t^*, y) + \beta F(y, x_{t+2}^*)\} \tag{1PP}$$

subject to

$$\begin{aligned} y &\in \Gamma(x_t^*) \\ x_{t+2}^* &\in \Gamma(y) \end{aligned}$$

If $y \neq x_{t+1}^*$, this would contradict that the original sequence was optimal in (SP)[1], as we would had found a one period deviation (keeping everything else constant) that would yield a higher present discounted value of returns. Thus the minimal requirement is that x_{t+1}^* be *at least one* of the solutions for the above problem.

[1]This is if x_{t+1}^* is unique. Otherwise, read the statement as 'if $y \notin \arg\max(SP)$, subject to the constraints'.

Proposition 2.4.2. *Let (E1), (E2), (M1), (M2), (D) hold. Then, the necessary first-order conditions for (SP) coincide with those of (1PP) for all $t \geq 0$, that is:*

$$\begin{aligned} 0 &= F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) \\ x_{t+1}^* &\in \Gamma(x_t^*) \\ x_{t+2}^* &\in \Gamma(x_{t+1}^*) \end{aligned}$$

Thus under rather reasonable assumptions, the Euler Equation is a necessary condition for optimality in the sequence problem. It is, therefore, a necessary condition for optimality in any recursive formulation that is based on the sequence problem.

It is interesting to note that, for example, concavity of the return function or convexity of the constraint correspondence are not required for the Euler Equation to be necessary.

2.4.4 Necessity of the Transversality Condition

Consider a n-dimensional problem, that is $\dim(x_t) = n$. We have established that, for such a problem, n Euler Equations (EE) are necessary to trace out the optimal path. Note, however, that the EE form a system of n second-order difference equations. This means that we need $2n$ **boundary conditions** to completely characterise the optimal path: to see why this is true, look at the EE in the previous section - x_{t+1}^* is only determined if we have *both* x_t^* and x_{t+2}^*.
Half of the problem is solved by the fact that we take x_0, the initial states, as given. This takes care of n boundary conditions. However, if no further conditions are imposed, the solution may still be indeterminate. To solve this, we include **transversality conditions** (or *final conditions*) to pin down the optimal path. These are conditions of the type

$$\lim_{T \to \infty} \beta^T F_1(x_T^*, x_{T+1}^*) x_T^* = 0 \tag{TC}$$

Intuitively, TC tells us that the present discounted value of the state should go to zero as the time horizon expands. This condition stems very naturally from finite horizon reasoning: the state is valuable, and the more I have of it, the happier I am. The whole point of the problem is to manage the state so that it 'does not run out' and I can keep extracting happiness from it while I am alive. Therefore, when I die at $t = T$, I want to completely exhaust the state (but I really only want that to happen *at that specific point*). If the value of the state is not zero when I die, then I could change my policy and improve my situation. Thus, as long as I value the state, that is $F_1 > 0$, it is optimal to have $x_T^* = 0$. The only instance in which I would be willing to allow $x_T^* > 0$ is when the state is worthless to me, or $F_1 = 0$. Thus, (TC) is, in some sense, analogous to a dynamic Kuhn-Tucker Complementary Slackness Condition: either the state is valuable so its final amount should be driven to zero, or it is not valuable and in that case the program can end with a positive amount of it.

We proceed with some definitions that are useful for studying when is (TC) necessary.

Definition 2.4.1. *A path $\{x_t\}_{t=0}^{\infty}$ is **feasible** if, given x_0, $x_{t+1} \in \Gamma(x_t), \forall t \geq 0$.*

Assumption 2.4.1. *Any feasible path $\{x_t\}_{t=0}^{\infty}$ can be evaluated. That is*

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \in \mathbb{R}$$

Definition 2.4.2. *A feasible path $\{x_t^*\}_{t=0}^{\infty}$ is **optimal** if, for any $\{x_t\}_{t=0}^{\infty}$ feasible*

$$\sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

Definition 2.4.3. *A feasible path $\{x_t\}_{t=0}^{\infty}$ is **interior** if $x_{t+1} \in int[\Gamma(x_t)], \forall t \geq 0$ (note that this is silent on x_0).*

These definitions and assumptions take us to the main course:

Theorem 2.4.1. *(Kamihigashi, Economic Theory 2002) - Assume (M1), (M2), (C1), (C2), (D). Assume that $0 \in \Gamma(0), F_2(x, y) \leq 0$. Let $\{x_t^*\}_{t=0}^{\infty}$ be any interior optimal path. Then:*

$$\lim_{T\to\infty} [-\beta^T F_2(x_T^*, x_{T+1}^*) x_{T+1}^*] = 0$$

Proof. For notational simplicity, let $x^* := \{x_t^*\}_{t=0}^{\infty}$. The proof relies on the following lemma:

Lemma 2.4.1. *Let $f : [0, 1] \to \mathbb{R} \cup \{-\infty\}$ be concave with $f(1) > -\infty$. Then, $\forall \gamma \in [0, 1), \forall \lambda \in [\gamma, 1)$ we have that*

$$\frac{f(1) - f(\lambda)}{1 - \lambda} \leq \frac{f(1) - f(\gamma)}{1 - \gamma}$$

Let $\lambda \in [0, 1)$ and consider the following path, based on the original optimal one:

$$x^\lambda = \{x_0^*, x_1^*, \ldots, x_T^*, \lambda x_{T+1}^*, \lambda x_{T+2}^*, \ldots\}$$

That is, we 'slightly contract' the original optimal path from $T + 1$ onwards. This new path is still feasible for λ close to 1, as x^* was interior and the constraint correspondence is monotonic by (M2). From optimality of x^*, x^λ cannot yield a strictly greater payoff. By assumption, all feasible paths can be evaluated, hence we must have that

$$\sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) \geq \sum_{t=0}^{\infty} \beta^t F(x_t^\lambda, x_{t+1}^\lambda)$$

However, we know that the paths coincide until period T, hence we can cancel all those terms and rearrange so to be left with

$$\beta^T [F(x_T^*, \lambda x_{T+1}^*) - F(x_T^*, x_{T+1}^*)] \leq \sum_{t=T}^{\infty} \beta^{t+1} [F(x_{t+1}^*, x_{t+2}^*) - F(\lambda x_{t+1}^*, \lambda x_{t+2}^*)]$$

We can interpret the LHS as the gains from a one period deviation, and the RHS as the subsequent losses that result from that deviation. The above expression then tells us that any incurred losses exceed one period gains from deviating from an optimal path. Divide both sides by $1-\lambda$

$$\frac{\beta^T[F(x_T^*, \lambda x_{T+1}^*) - F(x_T^*, x_{T+1}^*)]}{1-\lambda} \leq \frac{\sum_{t=T}^{\infty} \beta^{t+1}[F(x_{t+1}^*, x_{t+2}^*) - F(\lambda x_{t+1}^*, \lambda x_{t+2}^*)]}{1-\lambda}$$

Now, consider some $\gamma \in (0, \lambda)$. If we replace all λ on the RHS by γ, we will be making the difference even wider. This is because $\gamma < \lambda$ and, by assumption, F is monotonic. Therefore:

$$\frac{\beta^T[F(x_T^*, \lambda x_{T+1}^*) - F(x_T^*, x_{T+1}^*)]}{1-\lambda} \leq \frac{\sum_{t=T}^{\infty} \beta^{t+1}[F(x_{t+1}^*, x_{t+2}^*) - F(\gamma x_{t+1}^*, \gamma x_{t+2}^*)]}{1-\gamma}$$

Now, let $\gamma \to 0$, and divide and multiply by x_{T+1}^* on the LHS to obtain:

$$\frac{\beta^T[F(x_T^*, \lambda x_{T+1}^*) - F(x_T^*, x_{T+1}^*)]x_{T+1}^*}{(1-\lambda)x_{T+1}^*} \leq \sum_{t=T}^{\infty} \beta^{t+1}[F(x_{t+1}^*, x_{t+2}^*) - F(0,0)]$$

Note that the sequence $x_t = 0, \forall t \geq 0$ is feasible as we assumed that $0 \in \Gamma(0)$. Further note that we can rewrite the LHS as

$$\frac{\beta^T[F(x_T^*, x_{T+1}^* - (1-\lambda)x_{T+1}^*) - F(x_T^*, x_{T+1}^*)]x_{T+1}^*}{(1-\lambda)x_{T+1}^*}$$

Factoring out $\beta^T x_{T+1}^*$ and letting $\lambda \to 1$ leaves us with the derivative of F with respect to its second argument!

$$\beta^T x_{T+1}^* \lim_{\lambda \to 1} \frac{F(x_T^*, x_{T+1}^* - (1-\lambda)x_{T+1}^*) - F(x_T^*, x_{T+1}^*)}{(1-\lambda)x_{T+1}^*} = -\beta^T x_{T+1}^* F_2(x_T^*, x_{T+1}^*)$$

Now, by assumption, $F_2 \leq 0$, hence we have that

$$0 \leq -\beta^T x_{T+1}^* F_2(x_T^*, x_{T+1}^*) \leq \sum_{t=T}^{\infty} \beta^{t+1}[F(x_{t+1}^*, x_{t+2}^*) - F(0,0)]$$

We know, by assumption, that any feasible sequence can be evaluated. This then means that the series of payoffs generated by any feasible sequence must converge, hence its tail converges to zero. That is, as $T \to \infty$ the RHS of the above inequality must converge to zero. Taking limits leaves us then with

$$0 \leq \lim_{T \to \infty} [-\beta^T x_{T+1}^* F_2(x_T^*, x_{T+1}^*)] \leq 0$$

Thus we have proved that optimality implies the above condition or, put it differently, that the above condition is necessary for a path to be optimal. □

The (TC) that we have just worked with, and whose necessity we have just shown, is stated in slightly different terms than the (TC) that was originally presented. While the original (TC) was written in terms of the derivative with respect to state, F_1, the above one includes the derivative with respect to the control, F_2. This equivalence between the two stems from the Euler Equation:

$$F_2(x_t^*, x_{t+1}^*) = -\beta F_1(x_{t+1}^*, x_{t+2}^*)$$

Allowing us then two write

$$\lim_{T\to\infty}[-\beta^T x_{T+1}^* F_2(x_T^*, x_{T+1}^*)] = \lim_{T\to\infty}[\beta^{T+1} x_{T+1}^* F_1(x_{T+1}^*, x_{T+2}^*)] = 0$$

which is a slightly more common formulation of the TC.

2.4.5 Sufficiency of the Euler and Transversality Conditions

In general, as we have seen, both (EE) and (TC) will be necessary for most problems we encounter. However, if the problem is sufficiently more behaved (under further restrictions), they can actually turn out to be (jointly) sufficient for optimality. The following theorem addresses this issue

Theorem 2.4.2. *(Stokey & Lucas, 4.15) Let $X \subset \mathbb{R}_+^n$ and F satisfy (E1), (E2), (M1), (C1), (D). Then, the sequence x^* with $x_{t+1}^* \in int[\Gamma(x_t^*)]$ is optimal for (SP), given x_0, if it satisfies (EE) and (TC).*

Proof. Let x^* be a feasible sequence given x_0 that satisfies (EE) and (TC). We want to show that it is optimal, hence it is sufficient to show that the difference between the value of x^* and the value of any other sequence x is non-negative. Assuming that both sequences can be evaluated, we want to analyse the sign of

$$D := \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F(x_t^*, x_{t+1}^*) - F(x_t, x_{t+1})]$$

From (C1) and (D), F is strictly concave and differentiable at any $x_{t+1} \in \text{int}[\Gamma(x_t)]$, its value at any (x_t, x_{t+1}) is lower than the value of its first order Taylor Expansion. That is, expanding around (x_t^*, x_{t+1}^*):

$$F(x_t, x_{t+1}) \leq F(x_t^*, x_{t+1}^*) + F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*) + F_2(x_t^*, x_{t+1}^*)(x_{t+1} - x_{t+1}^*)$$

Rearranging:

$$F(x_t^*, x_{t+1}^*) - F(x_t, x_{t+1}) \geq F_1(x_t^*, x_{t+1}^*)(x_t^* - x_t) + F_2(x_t^*, x_{t+1}^*)(x_{t+1}^* - x_{t+1})$$

Multiplying both sides by β^t and summing over t, this allows us to establish that

$$D \geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F_1(x_t^*, x_{t+1}^*)(x_t^* - x_t) + F_2(x_t^*, x_{t+1}^*)(x_{t+1}^* - x_{t+1})]$$

Now, notice that the initial condition is fixed for any feasible sequence, $x_0^* = x_0$. Therefore, rearranging terms:

$$D \geq \lim_{T\to\infty} \left\{ \sum_{t=0}^{T-1} \beta^t [F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*)](x_{t+1}^* - x_{t+1}) + \beta^T F_2(x_T^*, x_{T+1}^*)(x_{T+1}^* - x_{T+1}) \right\}$$

From the assumption that x^* satisfies (EE), the summation is equal to zero for any t. Hence we are left with

$$D \geq \lim_{T\to\infty} \beta^T F_2(x_T^*, x_{T+1}^*)(x_{T+1}^* - x_{T+1})$$

We can use, once again, (EE) to replace F_2 with $-\beta F_1$ and get

$$D \geq -\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)(x_{T+1}^* - x_{T+1}) = -\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)x_{T+1}^* + \lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)x_{T+1}$$

By assumption (M1), F is increasing on its first argument, hence $F_1 > 0$. We have also assumed that $X \subseteq \mathbb{R}_+^n \Rightarrow x_t \geq 0, \forall t \geq 0$, hence we can get rid of the second term to widen the inequality

$$D \geq -\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)x_{T+1}^* + \lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)x_{T+1} \geq -\lim_{T\to\infty} \beta^T F_1(x_T^*, x_{T+1}^*)x_{T+1}^*$$

But since x^* satisfies (TC), the RHS of the inequality is equal to zero. Hence $D \geq 0$ and we conclude that if a sequence satisfies (EE) and (TC), then it must yield a payoff in present discounted value greater than any other feasible sequence. Therefore, it must be optimal for the (SP). □

Note that we could achieve sufficiency of (EE) and (TC) with very few restrictions on the constraint correspondence Γ (only that it is nonempty, compact and continuous). We did not use, for example, neither monotonicity or convexity. However, the result presupposes that a candidate optimal sequence x^* has been located and is feasible. For such a sequence to exist (one that satisfies EE and TC), further restrictions on Γ may need to be imposed.

2.4.6 Lucas 1978 Revisited

Did Lucas miss any viable equilibria by imposing the (seemingly) exogenous no bubble condition when solving for the tree price? We can now put our knowledge at work and confirm that this is not the case. Focusing in the case with no uncertainty (the results are easily adapted to account for that), recall that the sequence problem could be written as

$$\max_{\{c_t, s_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$\begin{aligned} c_t + \phi_t s_{t+1} &= (\phi_t + d_t) s_t \\ s_{t+1}, c_t &\geq 0 \\ s_0 &= 1 \end{aligned}$$

Let us modify the problem by eliminating the budget constraint and leaving everything in terms of an endogenous state variable which is also the control - s_t

$$\max_{\{s_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U[(\phi_t + d_t)s_t - \phi_t s_{t+1}]$$

subject to

$$\begin{aligned} s_{t+1} &\in \Gamma(s_t, d_t) = \left[0, \frac{1}{\phi_t}(\phi_t + d_t)s_t\right] \\ s_0 &= 1 \end{aligned}$$

We can easily confirm that this problem satisfies all the assumptions we require for both necessity and sufficiency of (EE) and (TC):

1. (E1) In this case, $X = \mathbb{R}_+^n$ and $\Gamma(s_t, d_t)$ is clearly nonempty (if we assume $d_t > 0$ for example), compact-valued and continuous.
2. (E2) $F = U(s_t, s_{t+1})$ is assumed to be bounded and continuous in the 1978 paper, the discount factor satisfies $\beta \in (0, 1)$.
3. (M1) Given that U is strictly increasing on its (single) argument, it is clearly increasing in s_t.
4. (M2) Γ is clearly monotonic: $s'_t \geq s_t$ implies that $\Gamma(s_t, d_t) \subseteq \Gamma(s'_t, d_t)$. It is also monotonic on d_t.
5. (C1) Utility is assumed to be strictly concave.
6. (C2) Γ is clearly convex valued: if $s_{t+1} \leq \frac{1}{\phi_t}(\phi_t + d_t)s_t$ and $s'_{t+1} \leq \frac{1}{\phi_t}(\phi_t + d_t)s'_t$, then for any $\lambda \in [0, 1]$ we obtain

$$\lambda s_{t+1} + (1-\lambda)s'_{t+1} \leq \lambda \frac{1}{\phi_t}(\phi_t + d_t)s_t + (1-\lambda)\frac{1}{\phi_t}(\phi_t + d_t)s'_t$$

7. (D) Utility is assumed to be twice differentiable.
8. (I) Follows immediately from the fact that utility is strictly increasing.

Therefore, (EE) and (TC) should be both necessary and sufficient. Considering market clearing for goods, $c_t = d_t$, we can write (EE) as

$$U'(d_t)\phi_t = \beta U'(d_{t+1})(\phi_{t+1} + d_{t+1})$$

and (TC) can be written in its F_1 form as

$$\lim_{T\to\infty} \beta^T U'(d_T)(\phi_T + d_T)s_T = 0$$

In equilibrium, $s_t = 1, \forall t \geq 0$, hence (TC) becomes

$$\lim_{T\to\infty} \beta^T U'(d_T)(\phi_T + d_T) = \lim_{T\to\infty} \beta^T U'(d_T)\phi_T + \lim_{T\to\infty} \beta^T U'(d_T)d_T = 0$$

Now recall that in the original paper, it was argued that $U'(d_t)d_t$ was bounded. This means that the second term is equal to zero, from $\beta < 1$. So the final form of the (TC) becomes

$$\lim_{T\to\infty} \beta^T U'(d_T)\phi_T = 0$$

Now, to obtain the asset price, solve for ϕ_t using (EE)

$$\phi_t \left[1 - \beta\frac{U'(d_{t+1})}{U'(d_t)}L^{-1}\right] = \beta\frac{U'(d_{t+1})}{U'(d_t)}d_{t+1}$$

where L^{-1} is the forward lag operator. This gives rise to the following general solution for some $k \in \mathbb{R}$ (that is, not assuming bubbles away):

$$\phi_t = \sum_{j=1}^{\infty} \beta^j \frac{U'(d_{t+j})}{U'(d_t)} d_{t+j} + k\frac{\beta^{-t}}{U'(d_t)}$$

To see that this solution is valid, note that

$$k\frac{\beta^{-t}}{U'(d_t)}\left[1 - \beta\frac{U'(d_{t+1})}{U'(d_t)}L^{-1}\right] = k\frac{\beta^{-t}}{U'(d_t)} - \beta\frac{U'(d_{t+1})}{U'(d_t)}k\frac{\beta^{-(t+1)}}{U'(d_{t+1})} = k\frac{\beta^{-t}}{U'(d_t)} - k\frac{\beta^{-t}}{U'(d_t)} = 0$$

Now, the asset pricing equation should satisfy (TC). Plugging the equation we have found on the expression describing (TC):

$$\lim_{T\to\infty} \beta^T U'(d_T)\left[\sum_{j=1}^{\infty} \beta^j \frac{U'(d_{T+j})}{U'(d_T)} d_{T+j} + k\frac{\beta^{-T}}{U'(d_T)}\right] = 0$$

Rearranging leaves us with

$$\lim_{T\to\infty} \beta^T \sum_{j=1}^{\infty} \beta^j U'(d_{T+j})d_{T+j} + k = 0$$

Since $U'(d_{T+j})d_{T+j}$ is bounded, the first term goes to zero. Is this so obvious? Note that $U'(d_{T+j})d_{T+j} \leq B, \forall t \geq 0$, and since marginal utility, the dividends and the discount factor are all positive terms:

$$0 \leq \lim_{T\to\infty} \beta^T \sum_{j=1}^{\infty} \beta^j U'(d_{T+j})d_{T+j} \leq \lim_{T\to\infty} \beta^T \sum_{j=1}^{\infty} \beta^j B = \lim_{T\to\infty} \beta^T \frac{B}{1-\beta} = 0$$

But then we are left with

$$\lim_{T\to\infty} k = 0 \Leftrightarrow k = 0$$

This means that the only possible value that k can take is zero, in order for (TC) (which is both necessary and sufficient) to be satisfied. Therefore, Lucas was not - indeed - missing any solution or equilibrium by assuming away bubbles!

Chapter 3

Search

3.1 Introduction

This section deals with Models of Search, widely used in fields such as labour economics and monetary theory. We start with some mathematical preliminaries, then proceed to partial equilibrium models of job search and conclude by looking into full-fledged general equilibrium models of search. The key feature of these models is the direct formalisation of the mechanics of trade.

Walrasian equilibrium models assume away, or take as given, the details of the trading process. This is usually achieved by postulating the existence of an auctioneer and/or a clearing house, where a single price is set/arises and all agents conduct their desired trades simultaneously, at that given price.

Search models, on the other hand, do not assume that trade is, any way, centralised: each individual autonomously 'searches' for trade opportunities, and the trading process, if a match is found, is concluded in an isolated manner. Thus, even though models of search are rather silent on what concerns the definition and the mechanisms that underlie *equilibrium prices*, this brief presentation evidences the fact that *several* prices may coexist in equilibrium due to the decentralised nature of the trade process.

3.2 Mathematical Preliminaries - Poisson Processes

3.2.1 Some Definitions

We start by providing some basic definitions concerning stochastic processes.

Definition 3.2.1. *A **random variable** is a measurable function.*

Definition 3.2.2. *A **stochastic process** is a sequence of random variables, $X = \{x(t) : t \in \mathcal{T}_+\}$, where $\mathcal{T}_+$ represents the time dimension. If time is discrete, $\mathcal{T}_+ = \mathbb{N} \cup \{0\}$. If time is continuous, $\mathcal{T}_+ = \mathbb{R}_+$.*

Definition 3.2.3. *A continuous-time stochastic process is said to have **independent increments** if for all $t_0, t_1, \ldots, t_n$, the random variables*

$$x(t_1) - x(t_0), x(t_2) - x(t_1), \ldots, x(t_n) - x(t_{n-1})$$

are independent.

Definition 3.2.4. *A stochastic process is said to have **stationary increments** if the distribution of $x(t+s) - x(t)$ is independent of t.*

That is, the distribution of the increments should depend only on their *size* (time-wise), and not on their location.

Definition 3.2.5. *A continuous-time stochastic process $\{N(t) : t \geq 0\}$ is a **counting process** if $N(t)$ denotes the number of events/arrivals that occurred on $(0, t]$. A counting process must satisfy the following properties:*

1. *$N(t) \geq 0$, there cannot be a negative number of events.*
2. *$N(t) \in \mathbb{N} \cup \{0\}$, must be integer valued (there cannot be 'half' of an event).*
3. *$N(s) \leq N(t)$ for all $s \leq t$ (events do not 'disappear').*
4. *For $s < t$, $N(t) - N(s)$ must denote the number of arrivals on $(s, t]$.*

Definition 3.2.6. *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be $o(\Delta)$ if*

$$\lim_{\Delta \to 0} \frac{f(\Delta)}{\Delta} = 0$$

The following two are the most important definitions of this section:

Definition 3.2.7. *(Poisson Process A) The counting process $\{N(t) : t \geq 0\}$ is said to be a **Poisson Process** with arrival rate $\alpha > 0$ if*

1. *$N(0) = 0$*
2. *The process has independent and stationary increments*
3. *As $\Delta \to 0$*
   (a) $\Pr[N(\Delta) = 1] = \alpha\Delta + o(\Delta)$
   (b) $\Pr[N(\Delta) = 0] = 1 - \alpha\Delta + o(\Delta)$
   (c) $\Pr[N(\Delta) \geq 2] = o(\Delta)$

It should be remarked that α *is not a probability*, but rather a rate of arrival. That is, as $\Delta \to 0$, then $\alpha\Delta$ does indeed constitute a probability, but it should not be generally interpreted as such (it can exceed one, for example).

Definition 3.2.8. *(Poisson Process B) The counting process $\{N(t) : t \geq 0\}$ is said to be a* ***Poisson Process*** *with arrival rate $\alpha > 0$ if*

1. $N(0) = 0$
2. *The process has independent increments*
3. *The number of arrivals in an interval of length Δ follows a Poisson distribution with mean $\alpha\Delta$. That is, for any $\Delta \geq 0$ we have that*

$$\Pr[N(t+\Delta) - N(t) = m] = \frac{(\alpha\Delta)^m e^{-\alpha\Delta}}{m!}$$

Note that condition 2) no longer requires stationary increments. It turns out that this is embedded in the 'new' condition 3).

3.2.2 A Big Theorem

The following result should be expected:

Theorem 3.2.1. *Definitions 3.7 (A) and 3.8 (B) are equivalent.*

Proof. We first show that $A \Rightarrow B$. The first two conditions for B are direct and explicitly stated by A. Therefore, we just need to prove B.3). By stationarity of the increments, location is irrelevant. Thus we can set $t = 0$ without loss of generality. From the fact that $N(0) = 0$, we would like to obtain:

$$\Pr[N(\Delta) = m] = \frac{(\alpha\Delta)^m e^{-\alpha\Delta}}{m!}$$

Take some arbitrary interval $[0, t+h]$ and denote the probability of zero arrivals in that interval by $P_0(t+h) := \Pr[N(t+h) = 0]$. Note that we can decompose this probability in the intervals $[0, t]$ and $(t, t+h]$ by using the properties of the Poisson process and write

$$P_0(t+h) = \Pr[N(t) = 0 \wedge N(t+h) - N(t) = 0]$$

From the fact that increments are independent, the above probability is equal to the product of the probabilities of each event

$$P_0(t+h) = \Pr[N(t) = 0] \Pr[N(t+h) - N(t) = 0]$$

Once again, location is irrelevant by stationarity of the increments. This allows us to set $t = 0$ in the second term to obtain

$$P_0(t+h) = \Pr[N(t) = 0] \Pr[N(h) = 0] = P_0(t)P_0(h)$$

As we are going to make $h \to 0$, note that by definition A:

$$P_0(h) = 1 - \alpha h + o(h)$$

Thus we can rearrange the above expression to get

$$\begin{aligned} P_0(t+h) - P_0(t) &= P_0(t)[P_0(h) - 1] \\ \Leftrightarrow \frac{P_0(t+h) - P_0(t)}{h} &= -\alpha P_0(t) + \frac{o(h)}{h} P_0(t) \end{aligned}$$

Taking $h \to 0$, and noting that the LHS turns into a time derivative, we obtain

$$\dot{P}_0(t) = -\alpha P_0(t)$$

We can then solve the differential equation

$$\int_0^t \frac{\dot{P}_0(s)}{P_0(s)} ds = -\alpha[t - 0]$$

which yields

$$P_0(t) = P_0(0)e^{-\alpha t}$$

From the fact that $N(0) = 0$, we immediately obtain that $P_0(0) = 1$, thus

$$P_0(t) = e^{-\alpha t}$$

which is precisely the expression of the Poisson density for $m = 0$.

We now show that the Poisson density is also obtained for $m \geq 1$. In particular, we are interested in computing

$$\begin{aligned} P_m(t+h) &= \Pr[N(t+h) = m] \\ &= \sum_{k=0}^{m} \Pr[N(t) = m - k \wedge N(t+h) - N(t) = k] \\ &= \sum_{k=0}^{m} \Pr[N(t) = m - k] \Pr[N(t+h) - N(t) = k] \\ &= \sum_{k=0}^{m} \Pr[N(t) = m - k] \Pr[N(h) = k] \end{aligned}$$

where the last two lines follow from independence and stationarity, respectively, as we have done with $m = 0$. By assumption, we only need, in the limit as $h \to 0$, to worry about the probabilities of events concerning zero and one arrivals. This is because the probability of more than one arrival in a very short time period is a term of order $o(\Delta)$. Separate these in the summation:

$$\sum_{k=0}^{m} \Pr[N(t)=m-k]\Pr[N(h)=k] = \Pr[N(t)=m]\Pr[N(h)=0] + \Pr[N(t)=m-1]\Pr[N(h)=1] + \sum_{k=2}^{m} \Pr[N(t)=m-k]\Pr[N(h)=k]$$

Note that the first term on the RHS is precisely equal to $P_m(t)$, while the second is $P_0(h)$ (which we have already computed). The third is $P_{m-1}(t)$ and the fourth is $P_1(h)$ (which we also know by assumption). All the terms in the sum are of order $o(h)$. Replacing for the definitions we then have that the above expression can be written as

$$P_m(t+h) = P_m(t)P_0(h) + P_{m-1}(t)P_1(h) + \sum_{k=2}^{m} \Pr[N(t)=m-k]\Pr[N(h)=k]$$

or, using what we know from definition A

$$P_m(t+h) = P_m(t)[1-\alpha h+\sigma(h)] + P_{m-1}(t)[\alpha h + o(h)] + \sum_{k=2}^{m} \Pr[N(t)=m-k]o(h)$$

Rearrange to obtain

$$\frac{P_m(t+h)-P_m(t)}{h} = -\alpha P_m(t) + P_m(t)\frac{o(h)}{h} + P_{m-1}(t)\alpha + P_{m-1}(t)\frac{o(h)}{h} + \sum_{k=2}^{m} \Pr[N(t)=m-k]\frac{o(h)}{h}$$

Taking $h \to 0$ the expression collapses to

$$\dot{P}_m(t) = \alpha[P_{m-1}(t) - P_m(t)]$$

We now proceed by using induction. For $m = 1$, we obtain

$$\dot{P}_1(t) = \alpha[P_0(t) - P_1(t)]$$

and we know that $P_0(t) = e^{-\alpha t}$, thus we can solve for $P_1(t)$ to obtain

$$P_1(t) = \alpha t e^{-\alpha t}$$

which, once again, coincides with the Poisson density for $m = 1$. The inductive step can now be performed: taking as given that

$$P_{m-1}(t) = \frac{(\alpha t)^{m-1}e^{-\alpha t}}{(m-1)!}$$

it can be found that

$$P_m(t) = \frac{(\alpha t)^m e^{-\alpha t}}{m!}$$

Thus proving that $A \Rightarrow B$.

Now, for the converse. We need to show that B.3) implies A.3) and stationarity of the increments. The latter fact follows directly from the properties of the Poisson distribution since, note, different probabilities arise only for different *intervals* and location is irrelevant. Recall that for a function that is K times continuously differentiable on $[x_0, x]$ and $K + 1$ times on (x_0, x) we can write the following expansion around x_0

$$f(x) = f(x_0) + \sum_{t=1}^{K+1} \frac{f^{(t)}(x_0)}{t!}(x - x_0)^t + R_K(x)$$

where the remainder can be written as

$$R_K(x) = \frac{f^{(K+1)}(\delta)}{(K+1)!}(x - x_0)^{(K+1)}$$

for some $\delta \in (x_0, x)$. This means that we can write the probability of m arrivals on some time interval Δ as a first order expansion

$$P_m(\Delta) = P_m(0) + P_m'(0)\Delta + \wr_m(\Delta)$$

(basically, think of x_0 as $t = 0$ and x as Δ). The remainder is a second-order term

$$\wr_m(\Delta) = \frac{1}{2}P_m''(\delta)\Delta^2$$

for some $\delta \in (0, \Delta)$. Note that $\wr_m(\Delta)$ is a term of order $o(\Delta)$ as

$$\lim_{\Delta \to 0} \frac{\wr_m(\Delta)}{\Delta} = \lim_{\Delta \to 0} \frac{P_m''(\delta)}{2}\Delta = 0$$

So our objective will be to take advantage of the fact that we know the probabilities to follow a Poisson distribution and take first order expansions to get to the expressions that are implied by definition A.

Consider now the case of $m = 0$ (0 arrivals). We know, from definition B that $P_0(\Delta) = e^{-\alpha\Delta}$ (the Poisson density). We also know that $P_0(0) = 1$ and that $P_0'(\Delta) = -\alpha e^{-\alpha\Delta}$. Therefore, $P_0'(0) = -\alpha$. Replacing in our first order expansion, we then obtain

$$P_0(\Delta) = 1 - \alpha\Delta + o(\Delta)$$

where the remainder as simply been replaced by $o(\Delta)$, and this is exactly what definition A tells us that the probability of zero arrivals should be for Δ close to zero!

For $m = 1$, once again the Poisson density tells us that $P_1(\Delta) = \alpha\Delta e^{-\alpha\Delta}$. This implies that the derivative is $P_1'(\Delta) = \alpha e^{-\alpha\Delta} - \alpha^2\Delta e^{-\alpha\Delta} = \alpha e^{-\alpha\Delta}(1 - \alpha\Delta)$. Thus, evaluated at zero, we obtain

$$P_1(0) = 0$$
$$P_1'(0) = \alpha$$

Replacing in our first order expansion

$$P_1(\Delta) = 0 + \alpha\Delta + o(\Delta)$$

which is, once again, what we wanted.

Finally, for $m \geq 2$, we know that

$$\begin{aligned} P_m(\Delta) &= \frac{(\alpha\Delta)^m e^{-\alpha\Delta}}{m!} \\ P_m'(\Delta) &= \frac{(\alpha)^m e^{-\alpha\Delta}}{m!}\Delta^{m-1}(m - \alpha\Delta) \end{aligned}$$

This means that, evaluated at $\Delta = 0$ we obtain

$$\begin{aligned} P_m(0) &= 0 \\ P_m'(0) &= 0 \end{aligned}$$

Thus our first order expansion is simply equal to

$$P_m(\Delta) = o(\Delta)$$

which is what we wanted to show. □

3.2.3 Interarrival Times

One of the key properties of Poisson processes is that they induce a very well-behaved distribution for the length of time periods between arrivals. This is a random variable that strongly interests us, as its distribution allows us to compute important economic variables such as the mean time that a worker should wait for another job offer to arrive. Furthermore, there is a direct relationship between the Poisson parameter (which is the rate of arrival - how many events are expected to happen in a certain period of time) and the parameter that will characterise the distribution of interarrival times. Let $\{t_n : n = 1, 2, \ldots\}$ denote a sequence of interarrival times. Its distribution, in the case of a Poisson counting process, is presented by the following Theorem:

Theorem 3.2.2. *The interarrival times $\{x_n : n = 1, 2, \ldots\}$ of a Poisson process with parameter $\alpha > 0$ are identically distributed* ***exponential*** *random variables with mean $\frac{1}{\alpha}$.*

Proof. Consider x_1, the first arrival time. We want to know what is $\Pr(x_1 \leq t) = 1 - \Pr(x_1 > t)$ for some $t \in \mathbb{R}_+$. Think about what $\Pr(x_1 > t)$ means - it should be the probability that no arrival is registered in the interval $[0, t]$. Thus it is equivalent to our well-known $P_0(t)$ which, as we know, is equal to $e^{-\alpha t}$ (by the Poisson density). Thus we have that

$$\Pr(x_1 \leq t) = 1 - e^{-\alpha t}$$

which is precisely the cdf of an exponential distribution with parameter $\frac{1}{\alpha}$.

Now, consider x_2, we want to know the probability that $x_2 \leq t$ for some t, given that the first arrival took place at some period s. Note that

$$\Pr[x_2 \leq t | x_1 = s] = 1 - \Pr[x_2 > t | x_1 = s]$$

and think about what the second term means: we are trying to compute the probability that there was an arrival at period s, and then t units of time elapsed before another arrival took place. Therefore, it is equivalent to computing the probability of no arrivals between s and $t + s$, which we can do using our Poisson density:

$$\begin{aligned}\Pr[x_2 \leq t | x_1 = s] &= 1 - \Pr[x_2 > t | x_1 = s] \\ &= 1 - \Pr[N(t+s) - N(s) = 0] \\ &= 1 - \Pr[N(t) = 0] \\ &= 1 - P_0(t) = 1 - e^{-\alpha t}\end{aligned}$$

where the second line follows from independence of the increments and the third follows from stationarity. This can be easily repeated for any $n \geq 3$. □

We are now endowed with all the machinery we need to analyse basic search models.

3.3 One-Sided Job Search

We start by focusing on a *partial equilibrium* job search model: there is an agent who is looking for a job. Every period, there will be a probability (which can be exogenous or endogenous) of 'being matched with a firm' and receiving a job offer in the form of some (for now exogenous) wage. The worker then has the opportunity to choose whether to accept or reject the wage offer.

3.3.1 Environment

The **environment** is as follows:

1. We allow for a flexible temporal structure. For now, let us work with discrete time, but allow the length of each interval to be a variable Δ. Thus $\mathcal{T} = \{\Delta, 2\Delta, 3\Delta, \ldots\}$. Naturally, letting $\Delta \to 0$ brings us to a continuous time model.
2. Let $\beta(\Delta)$ be the discount factor, as a function of the length of the time interval that is being used. The agent seeks to maximise expected utility over an infinite horizon
$$U = \mathbb{E} \sum_{t=0}^{\infty} \beta(\Delta)^t y_t$$
where y_t is total income received in period t. Note that the agent is risk-neutral.

3. The worker starts unemployed at $t = 0$, and looking for a job.
4. While unemployed, the worker gets a random number of job offers in a period of length Δ.
5. Wage offers are iid draws from some distribution F. Let $\mathcal{W} \subseteq \mathbb{R}_+$ denote the support of F. Let $F(0) = 0$ and $F(\bar{w}) = 1$ (so that the support is bounded). This implies that $\mathbb{E}(w) < \infty$
6. The worker gets n job offers at the beginning of each period, and considers only the best one: $\omega = \{w_1, \ldots, w_n\}$. Let the distribution of the best job offer be denoted by $G(\omega, n) = \Pr(\omega \leq \hat{w}) = F(\hat{w})^n$.
7. While employed at wage w, the worker earns an instantaneous payoff equal to $w\Delta$ every period.
8. While unemployed, the worker receives an exogenous unemployment benefits $b\Delta$ every period.
9. For now, we shall assume that no quits are allowed, no offers can be recalled and there is no on-the-job search.

3.3.2 Value Functions

Consider an agent who has accepted a job offer at wage w. Under our assumptions of no quits and no on-the-job search, the value of the job today is equal to the wage that is received plus the discounted continuation value of the job:

$$V(w) = w\Delta + \beta(\Delta)V(w)$$

What about an agent that is unemployed? Things become slightly trickier: the agent receives the benefits today. However, at the end of the period, the agent may either receive no job offers (and move to the next period unemployed) or may receive an arbitrary number of positive job offers. In this case, the agent looks at the offer that yields the best wage and compares the value of accepting that offer to the value of remaining unemployed (and, say, wait for a better offer in the following period). Let $P_0(\Delta)$ denote the probability of no arrivals in a period of length Δ, using our previous notation, and let $P_m(\Delta)$ denote the probability of m arrivals. Formally, we can write the value of unemployment as

$$U = b\Delta + \beta(\Delta)\left[P_0(\Delta)U + \sum_{n=1}^{\infty} P_n(\Delta) \int_0^{\bar{w}} \max\{V(\omega), U\} dG(\omega, n)\right]$$

Note that the expected value is taken with respect to the maximum wage, ω, and to the number of offers that arrive, which is also a random variable. Also, U is not a function of wages, as it only depends on the wage in expectation.

This seemingly daunting problem becomes extremely tractable if we add a couple of assumptions:

Assumption 3.3.1. *The number of offers that arrive each period follows a Poisson process with arrival rate* $\alpha > 0$

Assumption 3.3.2. *The discount factor has the form* $\beta(\Delta) = e^{-r\Delta}$ *for some* $r > 0$ *(which is the discount rate).*

Note that Assumption 3.2 implies that

$$\begin{aligned}
\beta(0) &= 1 \\
\lim_{\Delta\to\infty} \beta(\Delta) &= 0 \\
\beta(\Delta) &\in [0,1], \forall \Delta \geq 0
\end{aligned}$$

What does it imply for the value of employment? Replacing for $\beta(\Delta)$ we get

$$\begin{aligned}
V(w) &= w\Delta + e^{-r\Delta}V(w) & & \\
V(w)(1-e^{-r\Delta}) &= w\Delta & & \text{divide both sides by } \frac{1}{\Delta} \\
V(w)\frac{1-e^{-r\Delta}}{\Delta} &= w & & \text{take } \lim_{\Delta\to 0} \\
rV(w) &= w & &
\end{aligned}$$

Thus in continuous time, $rV(w)$ can be seen as the *flow value of employment* (at wage w). The same expression could had been derived from the utility function: at time t, for an agent who is employed at that wage, the present discounted value of utility is

$$V(w) = \int_t^\infty e^{-r(s-t)} w ds = \frac{w}{r}$$

Now, turning to the value of unemployment, let us take advantage of what definition A tells us about the expressions for the probability of m arrivals: separate the probability of exactly one arrival (for which we have a different expression) from the probabilities of more than one arrival. Note that in the case of exactly one arrival, $G(\omega, 1) = F(\omega)$, the distribution for the maximum wage coincides with the distribution of a single offer. This gives us

$$\begin{aligned}
U = b\Delta + e^{-r\Delta}[(1-\alpha\Delta + o(\Delta))U + (\alpha\Delta + o(\Delta)) \int_0^{\bar{w}} \max\{V(\omega), U\} dF(\omega) \\
+ \sum_{n=2}^{\infty} o(\Delta) \int_0^{\bar{w}} \max\{V(\omega), U\} dG(\omega, n)]
\end{aligned}$$

Note that, by assumption, $\mathcal{W}$ is bounded. Hence the term $\int_0^{\bar{w}} \max\{V(\omega), U\} dG(\omega, n)$ is bounded as well. Furthermore, it is being multiplier by $o(\Delta)$, thus the product is of order $o(\Delta)$. Group all terms of this order (as the sum of terms of this order is also of the same order) to obtain

$$U = b\Delta + e^{-r\Delta}\left[(1-\alpha\Delta)U + \alpha\Delta \mathbb{E}_w \max\{V(w), U\} + o(\Delta)\right]$$

Now, separate the second term $(1 - \alpha\Delta)U = U - \alpha\Delta U$. Move the first U (multiplier by the discount factor) to the LHS and group the second with the expected value of the max (this can be done since U is not a function of w). This gives us

$$U(1 - e^{-r\Delta}) = b\Delta + e^{-r\Delta}\alpha\Delta\mathbb{E}_w \max\{V(w) - U, 0\} + o(\Delta)$$

Divide both sides by Δ

$$U\frac{1 - e^{-r\Delta}}{\Delta} = b + e^{-r\Delta}\alpha\mathbb{E}_w \max\{V(w) - U, 0\} + \frac{o(\Delta)}{\Delta}$$

and let $\Delta \to 0$ to obtain the final expression

$$rU = b + \alpha\mathbb{E}_w \max\{V(w) - U, 0\}$$

which tells us that the flow value of unemployment is equal to the benefit that is received plus the expected *capital gain* that may arise from exercising an option - the option of accepting a job offer. The expectation, in explicit form, is given by

$$rU = b + \alpha \int_0^{\bar{w}} \max\{V(w) - U, 0\} dF(w)$$

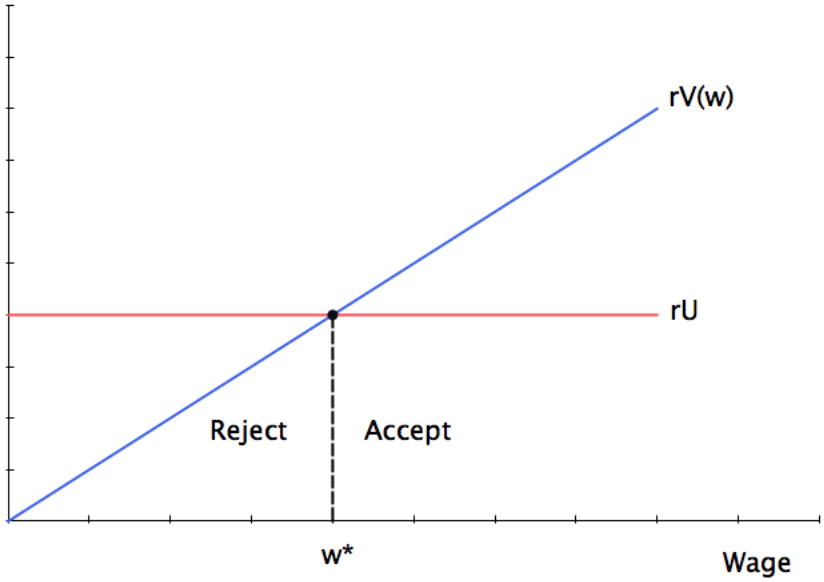
and we have seen before that

$$rV(w) = w$$

3.3.3 The Decision Rule

These two equations fully describe the behaviour of our job searcher. To see why this is the case, note that, as remarked before, U is not a function of w, whereas $V(w)$ is an increasing function of the wage. This induces a **reservation wage policy**. That is, the decision rule for the agent is to "accept an offer w whenever $V(w) \geq U$". Thus there exists w^* such that $V(w^*) = U$, and since the LHS is strictly increasing while the RHS is constant, the worker will accept any job offering $w \geq w^*$.

Figure 3.1: Decision Rule



This allows us to slightly simplify the expression for the value of unemployment: the capital gain is only exercised whenever $V(w) \geq U$, and this will only be the case when $w \geq w^*$. Therefore, we can adjust the lower bound of the integral and get rid of the max

$$rU = b + \alpha \int_{w^*}^{\bar{w}} (V(w) - U) dF(w)$$

We can extract some further insights from the model by using the following result:

Claim 3.3.1. $\int_{w^*}^{\hat{w}} (V(w) - U) dF(w) = \int_{w^*}^{\hat{w}} V'(w)[1 - F(w)] dw$

Proof. Take the original expression and integrate by parts

$$\begin{aligned}
\int_{w^*}^{\hat{w}} (V(w) - U) dF(w) &= [V(w) - U] F(w)|_{w^*}^{\hat{w}} - \int_{w^*}^{\hat{w}} V'(w) F(w) dw \quad && U \text{ does not depend on } w \\
&= [V(\hat{w}) - U] F(\hat{w}) - [V(w^*) - U] F(w^*) \\
&- \int_{w^*}^{\hat{w}} V'(w) F(w) dw \\
&= [V(\hat{w}) - U] F(\hat{w}) - \int_{w^*}^{\hat{w}} V'(w) F(w) dw \quad && \text{by definition } V(w^*) = U \\
&= \int_{w^*}^{\hat{w}} V'(w) dw F(\hat{w}) - \int_{w^*}^{\hat{w}} V'(w) F(w) dw \quad && \text{once again from } U = V(w^*) \\
&= \int_{w^*}^{\hat{w}} V'(w)[F(\hat{w}) - F(w)] dw
\end{aligned}$$

Setting $\hat{w} = \bar{w} \Rightarrow F(\hat{w}) = F(\bar{w}) = 1$ completes the proof. $\square$

This allows us then to write the value of unemployment as

$$rU = b + \alpha \int_{w^*}^{\bar{w}} V'(w)[1 - F(w)]dw$$

From the fact that

$$V'(w) = \frac{1}{r}$$

we then obtain

$$rU = b + \frac{\alpha}{r} \int_{w^*}^{\bar{w}} [1 - F(w)]dw$$

and note that we have managed to make the value function for unemployment independent from any other value function - it only depends now on the model's parameters and w^* (which is an endogenous variable). This might not seem a big deal: after all, we had explicitly computed the expression for $V(w)$ as a function of the parameters, so we could had simply replaced. It turns out, however, that in more complicated models it can become extremely difficult to explicitly compute $V(w)$, but its derivative can be easily computed as a function of the parameters. Hence the usefulness of this general result.

The model is fully characterised if we compute the only remaining endogenous variable: the reservation wage w^*. To do this, we simply take advantage of its definition: $rV(w^*) = rU$. Using what we have:

$$w^* = b + \frac{\alpha}{r} \int_{w^*}^{\bar{w}} [1 - F(w)]dw$$

Thus we have one (integral) equation on w^*, allowing us to solve for this variable as a function of the parameters.

3.3.4 Comparative Statics

The following result describes some of the comparative statics.

Proposition 3.3.1. $\frac{\partial w^*}{\partial b} > 0, \frac{\partial w^*}{\partial r} < 0, \frac{\partial w^*}{\partial \alpha} > 0$

Proof. These results follow from application of the Implicit Function Theorem to the equation that describes w^*. Define the homogeneous equation

$$F(w^*, \alpha, r, b) = b + \frac{\alpha}{r} \int_{w^*}^{\bar{w}} [1 - F(w)]dw - w^* = 0$$

The partial derivative of w^* with respect to some arbitrary parameter θ can then be computed as

$$\frac{\partial w^*}{\partial \theta} = -\frac{1}{\frac{\partial F}{\partial w^*}} \frac{\partial F}{\partial \theta}$$

In this case, we have that

$$\frac{\partial F}{\partial w^*} = -\frac{\alpha}{r}[1 - F(w^*)] - 1 < 0$$

Thus the sign of the partial derivatives of w^* with respect to any parameter will coincide with the sign of the partial derivative of F with respect to that parameter. This means we just need to check the sign of $\frac{\partial F}{\partial \theta}$. We then have that

$$\frac{\partial F}{\partial b} = 1 > 0$$
$$\frac{\partial F}{\partial r} = -\frac{\alpha}{r^2}\int_{w^*}^{\bar{w}} [1 - F(w)]dw < 0$$
$$\frac{\partial F}{\partial \alpha} = \frac{1}{r}\int_{w^*}^{\bar{w}} [1 - F(w)]dw > 0$$

which concludes our proof. □

The intuition for these results is the following:

1. When the unemployment benefits rises, the value of unemployment increases. Thus the outside option of not working becomes more valuable, making the worker demand a higher reservation wage.
2. When the discount rate increases, the worker values the future less and is more concerned with current payoffs. Therefore, he will be more willing to take a lower paying job today, and is less sensitive to the fact that he will have to stick with that (low paying job) forever. This is quite intuitive: a more patient agent is likely to reject more offers until he finds one that pays a high enough wage.
3. When the rate of arrival for job offers increases, the worker receives, on average, more offers per period. Therefore, he can afford to become 'pickier' in selecting the best offer. This leads to an increase in the reservation wage.

It is also interesting to look at the *rate of exit from unemployment*. Since the number of offers that arrive in each period is independent from the wage that is associated to those offers, the **hazard rate of unemployment** is equal to

$$H = \alpha[1 - F(w^*)]$$

That is, the rate of arrival of offers times the probability that they are accepted. How does this rate change with α? In principle, the answer to this question is ambiguous: on the one hand, the worker gets more offers each period, which would increase the chances of leaving unemployment. But, on the other, as we have seen, this also makes the worker become choosier.

Claim 3.3.2. $\frac{\partial H}{\partial \alpha} > 0$ *if F is log-concave, i.e., if* $\frac{d^2 \log F(x)}{dx^2} \leq 0$.

3.3.5 The Discount Factor

The model unravelled very nicely due to the specific functional form that we assumed for the discount factor, without any discussion. Are the functional forms we obtained for the value functions without loss of generality? It turns out that they are extremely robust to the choice of discount factor. Consider, for example, a more 'traditional' specification for the discount factor:

$$\beta(\Delta) = \frac{1}{1 + r\Delta}$$

The value of unemployment can be written, as before, as

$$U = \frac{1}{1 + r\Delta} \left[b\Delta + +(1 - \alpha\Delta)U + \alpha\Delta\mathbb{E}_w \max\{V(w), U\} + o(\Delta) \right]$$

with the caveat that now the instantaneous payoff b is also discounted. Multiply both sides by $1 + r\Delta$ and rearrange to obtain

$$U + r\Delta U = b\Delta + \alpha\Delta\mathbb{E}_w \max\{V(w), U\} + U - \alpha\Delta U + o(\Delta)$$

Rearrange, divide both sides by Δ and make it go to zero to obtain

$$rU = b + \alpha\mathbb{E}_w \max\{V(w) - U, 0\}$$

which is exactly the same we got before.

3.3.6 Example with two Poisson shocks

Suppose now that the worker can die (or leave the labour market) at any point in time. This shock follows a Poisson distribution with rate of arrival given by δ. As before, let us start by setting up the problem in discrete terms for a time period of arbitrary length Δ. This new shock is realised before the arrival of job offers (this assumption helps with the math but is, in fact, irrelevant when $\Delta \to 0$, any timing assumption yields the same results).

Let us work with the different specification of the discount factor, $\beta(\Delta) = \frac{1}{1+r\Delta}$ (once again, this is irrelevant when we proceed to continuous time, and yields the same results). Assume that once the worker dies, he gets 0 forever. Therefore, the 'value of death' is simply equal to zero

$$D = 0$$

If employed, the worker receives the wage payoff $w\Delta$. Then, with probability $\delta\Delta + o(\Delta)$, he dies and proceeds to D. With probability $1 - \delta\Delta + o(\Delta)$, he does not die and earns the continuation value of employment $V(w)$. We disregard the probability of more than one arrival of the 'death event' as a term of order $o(\Delta)$. Thus we can write the value of employment as

$$V(w) = \frac{1}{1 + r\Delta} \left[w\Delta + \delta\Delta.0 + (1 - \delta\Delta)V(w) + o(\Delta) \right]$$

Rearrange to obtain

$$r\Delta V(w) = w\Delta - \delta\Delta V(w) + o(w)$$

Divide by Δ and take limits to arrive at the final expression

$$(r+\delta)V(w) = w$$

Note that the rate of arrival of death, δ, now acts as a discount factor. It is as if the rate at which the agent discounts the future has increased. This is extremely intuitive: after all, there is now an event that makes the future be worth zero, with probability $\delta\Delta$. Therefore, the worker should value the future less.

What about the value of unemployment? The reasoning is the same, but now our timing assumption gains relevance: the worker only moves to the 'job choosing' stage with probability $1-\delta\Delta+o(\Delta)$. Once again, collapse all the terms concerning more than one arrival:

$$U = \frac{1}{1+r\Delta}\left[b\Delta + \delta\Delta.0 + (1-\delta\Delta)\left\{\alpha\Delta\mathbb{E}_w \max\{V(w), U\} + (1-\alpha\Delta)U\right\} + o(\Delta)\right]$$

Rearrange to get

$$(1+r\Delta)U = b\Delta + (1-\delta\Delta)\alpha\Delta\mathbb{E}_w \max\{V(w), U\} + (1-\delta\Delta)U - (1-\delta\Delta)\alpha\Delta U + o(\Delta)$$

Rearrange a bit further and divide everything by Δ

$$(r+\delta)U = b + (1-\delta\Delta)\alpha\mathbb{E}_w \max\{V(w)-U, 0\} + \frac{o(\Delta)}{\Delta}$$

Let $\Delta \to 0$ to obtain the final expression

$$(r+\delta)U = b + \alpha\mathbb{E}_w \max\{V(w)-U, 0\}$$

Once again, the only thing that changes with respect to the baseline model is the 'increment' on the discount factor. Thus the model is, essentially, the same. It is easy to see that the worker will, once more, follow a reservation wage policy (as U does not depend on w and $V(w)$ is strictly increasing on this variable). Thus $\exists w^*$ such that $V(w^*) = U$. This means we can rewrite the expression for the value of unemployment as

$$(r+\delta)U = b + \alpha\int_{w^*}^{\bar{w}} (V(w)-U)dF(w)$$

We can simplify it further by noting that $V'(w) = \frac{1}{r+\delta}$ and applying the result on Claim 3.1:

$$(r+\delta)U = b + \frac{\alpha}{r+\delta}\int_{w^*}^{\bar{w}} [1-F(w)]dw$$

Thus the reservation wage solves

$$\begin{aligned}(r+\delta)V(w^*) &= U \\ \Leftrightarrow w^* &= b + \frac{\alpha}{r+\delta}\int_{w^*}^{\bar{w}} [1-F(w)]dw\end{aligned}$$

and everything follows as before.

What if the two shocks were correlated? In principle this would have a decisive impact on features such as the *timing* of the shocks. It turns out that this is irrelevant. Even if shocks are correlated, the correlation terms are of order $o(\Delta)$, hence they disappear once we move from discrete to continuous time, $\Delta \to 0$. When dealing with an instantaneous interval of time, correlation (and, more generally, the timing of the shocks) becomes irrelevant.

3.4 Equilibrium Search

So far, we have been dealing with one-sided partial equilibrium search models: the worker receives an exogenous stream of offers each period, and each of these is associated to a wage that follows some exogenous distribution. We now proceed to make the matching and wage determination process endogenous. The seminal work in this field was undertaken by Peter Diamond with his 1982 'Coconut Model'. This was then successfully translated to a labour market general equilibrium model by Mortensen and Pissarides in 1994.

3.4.1 Diamond 1982 - Aggregate Demand Management in Search Equilibrium (Coconut Model)

Consider an island populated by agents who like eating coconuts. Each of them can climb a tree to get a coconut. However, an agent cannot open a coconut alone (or he cannot eat the coconuts taken by himself) - he needs another agent to do it. Similarly, he can only eat coconuts taken by other agents. Therefore, once he gets a coconut, he has to look/search for another agent to exchange coconuts with before consuming.

Environment

Formally, the environment is as follows:

1. Time is continuous, $\mathcal{T} = [0, \infty)$
2. There is a continuum of agents of measure 1
3. There is a single indivisible consumption good (the 'coconut')
4. The production technology is as follows: the arrival of production opportunities (or 'trees') is a Poisson process with rate a. Once the agent faces a production opportunity, he may either skip or execute it (climb the tree). If the agent executes,

he produces one unit of the good and pays a production cost $c \sim G(\cdot)$, where $supp(G) = (\underline{c}, \bar{c})$ and $\underline{c} \geq 0$ (this is the cost of climbing the tree)

5. Agents face an inventory constraint of one unit of the good (they cannot carry more than one coconut at the same time)

6. Agents cannot consume the good they produce

7. From the standpoint of an individual agent, trading opportunities (meetings with other agents) arrive from a Poisson process at rate b

8. Agents get utility u from consuming a coconut and discount the future at rate $r > 0$. They seeks to maximise the following utility function

$$U = \mathbb{E}_0 \sum_{i=1}^{\infty} e^{-rt_i} \{\mathbb{1}(t_i \in \mathcal{T}_u)u - \mathbb{1}(t_i \in \mathcal{T}_c)c\}$$

where $\mathcal{T}_u$ denotes the subsets of the time space where the agent eats a coconut, and $\mathcal{T}_c$ are all subsets of the time space where the agent climbs a tree (produces). Note that even though time is continuous, the model dynamics are *lumpy* in the sense that the way the agent extracts utility (or pays a cost) is discrete (only happens at certain instants). There is no such thing as a continuous flow of utility as in the previous model. The expectation operator is taken over the random dates of consumption/production and the random variable c (thus u is a known constant).

Assumption 3.4.1. $u > \underline{c}$

Obviously, without the previous assumption, no agent would ever produce coconuts. We further assume that there is no credit/IOU's in this economy. Unlike Arrow-Debreu Walrasian settings, in which agents can trade credit contracts, trade can only arise *physically* in this model: you can only trade if you give a coconut and you get one in return. There are no promises or futures contracts.

The Congestion Externality

In this model, we can divide agents in two main groups:

- The **Production Sector** consists of agents who are looking for production opportunities. These can be seen as 'unemployed agents'.
- The **Exchange Sector** consists of agents carrying coconuts and looking for someone to trade with. These are 'employed agents'.

Let e denote the measure of agents in the exchange sector, so that there are $1 - e$ agents in the production sector. Let $b(e)$ be the Poisson rate of arrival of trading opportunities.

Assumption 3.4.2. *$b'(e) > 0$ and $b(0) = 0$*

The above assumption tells us that the more agents are there in the exchange sector, the easier (or the faster) it is to conclude a trade. Furthermore, it also tells us that if the measure of agents in the exchange sector is zero, then it is impossible to conclude a trade (as no matches arrive).

Assumption 3.4 is the heart of the model, in the sense that, as we will see, it is what makes the analysis of this model interesting in the first place. It represents a **congestion externality**: the more agents there are looking for trades, the easier it is conclude a trade. Even though it depends on the measure of agents wanting to trade, each agent will individually treat it as a parameter (as each agent is a measure zero atom). This already hints at potential inefficiencies that generate scope for government intervention.

The role of $b(e)$ is similar to that of a production function. It induces, in fact, a **matching function**. Let $m(e)$ denote the number of meetings ('matches') that take place between agents in the exchange sector at every instant. This can be measured as $eb(e)$ - that is, the total number of agents in the sector times the number of meetings that take place. Thus we have that

$$m(e) = eb(e) \Rightarrow b(e) = \frac{m(e)}{e}$$

As we would expect, the rate of arrival of matches is equal to the number of total matches divided by the number of agents that are looking for a match. Notice that if the matching function were homogeneous of degree one, $m(\lambda e) = \lambda m(e)$, then the matching rate...

$$b(\lambda e) = \frac{m(\lambda e)}{\lambda e} = b(e)$$

...has 'constant returns to scale'. This is only consistent with $b'(e) = 0$. This means that the assumption $b'(e) > 0$ is consistent with *increasing returns on the matching process*: an increase in the number of agents in the exchange sector makes the total number of matches increase more than proportionally, hence the externality.

One way to justify this assumption is by considering that there are, in fact, two islands: one where agents trade, and another where agents produce. Therefore, there are no *wasteful matches*: agents looking for a trade are never matched with agents who want to produce. Relaxing this assumption would be a way to eliminate the externality, as then the effective (productive) matching rate would not depend on the number of agents looking for a trade but rather on the total population of the economy, which is constant.

Value Functions

Now that the environment has been presented, let us proceed to analyse the model. As in the labour search environment, we construct the value function for 'employed' agents (those in the exchange sector) and for 'unemployed' (those looking for trees). Let $W_e(t)$ denote the value for an agent who is carrying a coconut and looking for a trade. Once again, we start by considering the discrete problem of an agent on a time interval $[t, t+\Delta]$ and then proceed by letting $\Delta \to 0$. Since the rate of arrivals is Poisson, we can ignore the probability of more than one arrival by time period as a $o(\Delta)$ term. Let $e^{-r\Delta}$ be the discount factor.

With probability $b\Delta$ the agent is matched with another one and may choose either to trade and get $u + W_u(t+\Delta)$, where W_u is the value of being unemployed (and looking for another coconut) or not to trade and keep looking, thus getting $W_e(t+\Delta)$. With probability $1 - b\Delta$, the agent does not find a match and keeps looking, getting the continuation value of remaining in the exchange sector $W_e(t+\Delta)$.

$$W_e(t) = e^{-r\Delta}\left\{b\Delta \max\{u + W_u(t+\Delta), W_e(t+\Delta)\} + (1-b\Delta)W_e(t+\Delta) + o(\Delta)\right\}$$

We assume, for now, that trading is always better than keeping searching, so that $\max\{u+W_u(t+\Delta), W_e(t+\Delta)\} = u+W_u(t+\Delta)$. This assumption will be justified shortly. Subtract $e^{-r\Delta}W_e(t)$ from both sides to get

$$W_e(t)(1-e^{-r\Delta}) = e^{-r\Delta}\left\{b\Delta[u + W_u(t+\Delta) - W_e(t)] + (1-b\Delta)[W_e(t+\Delta) - W_e(t)] + o(\Delta)\right\}$$

Divide everything by Δ

$$W_e(t)\frac{1-e^{-r\Delta}}{\Delta} = e^{-r\Delta}\left\{b[u + W_u(t+\Delta) - W_e(t)] + (1-b\Delta)\frac{W_e(t+\Delta) - W_e(t)}{\Delta} + \frac{o(\Delta)}{\Delta}\right\}$$

and notice that when we let $\Delta \to 0$, the second-to-last term becomes the time derivative of $W_e(t)$. Taking limits then leaves us with

$$rW_e(t) = b[u + W_u(t) - W_e(t)] + \dot{W}_e(t)$$

Once again, the above value function has an 'asset pricing interpretation': the LHS is the value of holding the asset for an additional instant (its value times the instantaneous interest rate). The RHS decomposes that value into a capital gain (the value of exercising the option of selling the asset) plus the time-change in the value of the asset.

For an agent in the production sector, with probability $a\Delta$ he finds a tree and learns of the cost of climbing it. He may either climb it, paying the cost and becoming employed $-c + W_e(t+\Delta)$, or he may keep looking for trees, earning $W_u(t+\Delta)$. With probability $1-a\Delta$, he does not find any tree and keeps looking, earning $W_u(t+\Delta)$. Thus the discrete value function is

$$W_u(t) = e^{-r\Delta}\left[a\Delta\mathbb{E}_c \max\{W_e(t+\Delta) - c, W_u(t+\Delta)\} + (1-a\Delta)W_u(t+\Delta) + o(\Delta)\right]$$

Subtract $e^{-r\Delta}W_u(t)$ from both sides

$$\begin{aligned} W_u(t)(1-e^{-r\Delta}) = e^{-r\Delta}[&a\Delta\mathbb{E}_c \max\{W_e(t+\Delta) - W_u(t) - c, W_u(t+\Delta) - W_u(t)\} \\ &+ (1-a\Delta)[W_u(t+\Delta) - W_u(t)] + o(\Delta)] \end{aligned}$$

Divide by Δ

$$\begin{aligned} W_u(t)\frac{1-e^{-r\Delta}}{\Delta} &= e^{-r\Delta}[a\mathbb{E}_c \max\{W_e(t+\Delta) - W_u(t) - c, W_u(t+\Delta) - W_u(t)\} \\ &\quad + (1-a\Delta)\frac{W_u(t+\Delta) - W_u(t)}{\Delta} + \frac{o(\Delta)}{\Delta}] \end{aligned}$$

and take limits. Notice that, once again, a time derivative arises in the second-to-last term. This leaves us with

$$rW_u(t) = a\mathbb{E}_c \max\{W_e(t) - W_u(t) - c, 0\} + \dot{W}_u(t)$$

The Decision Rule

Notice that the only decision that agents make is whether to climb the tree or not. For a given c, should I climb the tree I have just found or keep looking for a shorter one, whose cost of climbing is lower? Notice that neither $W_e(t)$ nor $W_u(t)$ depend on c (the latter only depends on its distribution). Clearly, from the expression for $W_u(t)$, the agent will only climb the tree if

$$W_e(t) - W_u(t) - c \geq 0$$

Thus, once again, we have a **reservation strategy**. That is, there exists a c^* such that

$$c^* = W_e(t) - W_u(t)$$

The agent will climb any trees such that $c \leq c^*$ and will not climb trees such that $c > c^*$, preferring to look for another tree instead. Notice that the problem is not necessarily stationary, hence the decision rule may be time-varying. Define then the function $c^*(t)$ to be given by

$$c^*(t) := W_e(t) - W_u(t)$$

Remark 3.4.1. *At this stage, we can justify our previous assumption that $u + W_u(t) \geq W_e(t)$. Notice that this is equivalent to $u \geq W_e(t) - W_u(t) = c^*(t)$. Clearly, it is never optimal for an agent to climb a tree whose cost exceeds the utility of trade.*

Equilibrium

Using the expressions we have found for the value functions, and subtracting one from the other

$$r[W_e(t) - W_u(t)] = b[u + W_u(t) - W_e(t)] - a\mathbb{E}_c \max\{W_e(t) - W_u(t) - c, 0\} + [\dot{W}_e(t) - \dot{W}_u(t)]$$

Replacing for $c^*(t)$ gives us

$$rc^*(t) = b[u - c^*(t)] - a\mathbb{E}_c \max\{c^*(t) - c, 0\} + \dot{c}^*(t)$$

or, writing the expectation explicitly, and noting that we can replace the max operator by restricting the upper bound of integration to $c^*(t)$, we obtain

$$rc^*(t) = b[u - c^*(t)] - a \int_{\underline{c}}^{c^*(t)} [c^*(t) - c]dG(c) + \dot{c}^*(t)$$

a differential equation on $c^*(t)$.

The model dynamics will be fully summarised by this dynamic policy $c^*(t)$ and its interaction with the aggregate state variable, $e(t)$. Clearly, $c^*(t)$ will crucially depend on b, which is in turn a function of $e(t)$. But this variable, the number of agents in the exchange sector, will depend on how many agents decide to climb trees, which in turn is determined by $c^*(t)$.

Let us think of the flows between sectors:

1. **Exchange $\rightarrow$ Production** - At every instant, there is a total of e agents in the exchange sector, who meet at rate $b(e)$. Therefore, a total of $eb(e)$ agents meet and undertake trades, moving to the production sector.

2. **Production $\rightarrow$ Exchange** - A total of $1-e$ agents 'meet trees' at rate a. However, the number of transitions is not equal to $a(1-e)$ as not all agents decide to climb the trees they meet. Agents only climb trees whose cost of climbing is lower than $c < c^*$. Thus the fraction of trees that are actually climbed is equal to $\Pr(c \leq c^*) = G(c^*)$. This means that a total of $a(1-e)G(c^*)$ agents move from the production to the exchange sector.

This suggests that the law of motion for people in the exchange sector is equal to arrivals minus departures, or

$$\dot{e}(t) = a[1 - e(t)]G[c^*(t)] - e(t)b[e(t)]$$

a differential equation for $e(t)$. This generates a system that fully describes the model's dynamics. This prepares us for defining an equilibrium for this model:

Definition 3.4.1. *A **Search Equilibrium** is a path $\{c^*(t), e(t)\}_{t\in\mathcal{T}}$ that satisfies*

$$rc^*(t) = b[e(t)][u - c^*(t)] - a\int_{\underline{c}}^{c^*(t)} [c^*(t) - c]dG(c) + \dot{c}^*(t) \tag{E1}$$

$$\dot{e}(t) = a[1 - e(t)]G[c^*(t)] - e(t)b[e(t)] \tag{E2}$$

Steady States and Comparative Statics

Our definition of equilibrium is extremely flexible, in the sense that it does not restrict the analysis of **steady states**.

Definition 3.4.2. *A steady state is a **stationary equilibrium**, a pair $\{c^*, e\}$ (a constant path) satisfying (E1) and (E2) and such that*

$$\dot{c}^*(t) = \dot{e}(t) = 0$$

Note that the set of steady states is a subset of the set of equilibrium paths. For simplicity, we now proceed to finding the steady states of the model. Take the equilibrium conditions and impose $\dot{c}^*(t) = \dot{e}(t) = 0$. This allows us to write the equilibrium conditions as

$$c^* = \frac{b(e)u - a\int_{\underline{c}}^{c^*}(c^* - c)dG(c)}{r + b(e)} \tag{SS1}$$

$$G(c^*) = \frac{eb(e)}{a(1-e)} \tag{SS2}$$

The steady state is obtained by solving this system of equations for (c^*, e). Note that the relationship between these two variables will depend on the properties of $b(e)$.

Proposition 3.4.1. *$\frac{\partial c^*}{\partial e}|_{\dot{c}(t)=0}$ is proportional to $b'(e)$.*

Proof. Given that this is the derivative of c^* with respect to e given stationary reservation costs, we can derive this result from (SS1) only. Multiply both sides by $r + b(e)$ and differentiate with respect to e to obtain

$$b'(e)c^* + [r + b(e)]\frac{\partial c^*}{\partial e} = b'(e)u - a\frac{\partial c^*}{\partial e}(c^* - c^*)g(c^*) - a\int_{\underline{c}}^{c^*} \frac{\partial c^*}{\partial e} dG(c)$$

Naturally, the second term on the RHS is equal to zero. Also note that $\frac{\partial c^*}{\partial e}$ does not depend on c, hence the last term is simply equal to $a\frac{\partial c^*}{\partial e}G(c^*)$, since $G(\underline{c}) = 0$. Rearranging for the derivative we are interested in leaves us then with

$$\frac{\partial c^*}{\partial e} = b'(e)\frac{u - c^*}{r + b(e) + aG(c^*)}$$

Since $u > c^*$, this derivative shares the same sign with $b'(e)$. □

Thus the way the reservation cost changes shares the sign with the derivative of the matching rate. Under our assumption of a positive congestion externality, $b' > 0$, agents will be willing to climb higher trees if there are more people to trade with: as the cost of trading (measured by the time that it takes to find a match) decreases. Basically, your investment pays off more quickly, thus people are more likely to invest in climbing trees.

Proposition 3.4.2. *$\frac{\partial^2 c^*}{\partial e^2}|_{\dot{c}(t)=0} < 0$ if $b''(e) < 0$.*

Proof. We found, in the proof of Proposition 3.2 that

$$b'(e)c^* + [r + b(e)]\frac{\partial c^*}{\partial e} = b'(e)u - aG(c^*)\frac{\partial c^*}{\partial e}$$

Differentiate again with respect to e to obtain

$$b''(e)c^* + b'(e)\frac{\partial c^*}{\partial e} + [r + b(e)]\frac{\partial^2 c^*}{\partial e^2} + b'(e)\frac{\partial c^*}{\partial e} = b''(e)u - aG(c^*)\frac{\partial^2 c^*}{\partial e^2} - ag(c^*)\left(\frac{\partial c^*}{\partial e}\right)^2$$

where g is the density associated with G. Rearrange to obtain

$$\frac{\partial^2 c^*}{\partial e^2} = \frac{b''(e)(u - c^*) - a\left(\frac{\partial c^*}{\partial e}\right)^2 g(c^*) - 2b'(e)\frac{\partial c^*}{\partial e}}{r + b(e) + aG(c^*)}$$

Clearly, $b''(e) < 0$ is sufficient for the above derivative to be negative. □

The logic is analogous to that of the previous proposition: if there are decreasing marginal returns over the matching rate, the same is true for the relationship between c^* and e.

Some comparative statics follow, for a stationary reservation cost

Proposition 3.4.3. *If $b'(e) > 0$, then*

- $\frac{\partial c^*}{\partial u}|_{\dot{c}(t)=0} > 0$
- $\frac{\partial c^*}{\partial a}|_{\dot{c}(t)=0} < 0$
- $\frac{\partial c^*}{\partial r}|_{\dot{c}(t)=0} < 0$

Proof. Analogous to that of Proposition 3.2. □

Intuition:

1. If the utility of trading increases, people are willing to climb higher trees. Not surprising.
2. If the rate of arrival of trees increases, people become 'choosier' with respect to the trees they climb. The logic is analogous to that of the increase in the rate of arrival of job offers - the reservation wage would increase.
3. If the discount rate goes up, people value trading less (as there is a time gap between producing and trading). Thus the effective payoff of climbing a trade went down, meaning that people will climb less trees.

If the number of agents in the trading sector is constant, on the other hand, we still obtain similar results:

Proposition 3.4.4. $\frac{\partial c^*}{\partial e}|_{\dot{e}(t)=0} > 0$ *if* $b'(e) > 0$

Proof. A stationary measure of agents in the exchange sector implies that we should work with (SS2). Adopt the same approach and totally differentiate the expression with respect to e, after multiplying both sides by $a(1-e)$

$$-aG(c^*) + a(1-e)g(c^*)\frac{\partial c^*}{\partial e} = b(e) + eb'(e)$$

Rearranging

$$\frac{\partial c^*}{\partial e} = \frac{b(e) + eb'(e) + aG(c^*)}{a(1-e)g(c^*)} > 0$$

□

Note that this condition is slightly weaker, as $b'(e) > 0$ is only sufficient, but not necessary (as it was before). The intuition is the same: more agents in the exchange sector → easier to trade → people climb more trees.

Dynamic Analysis

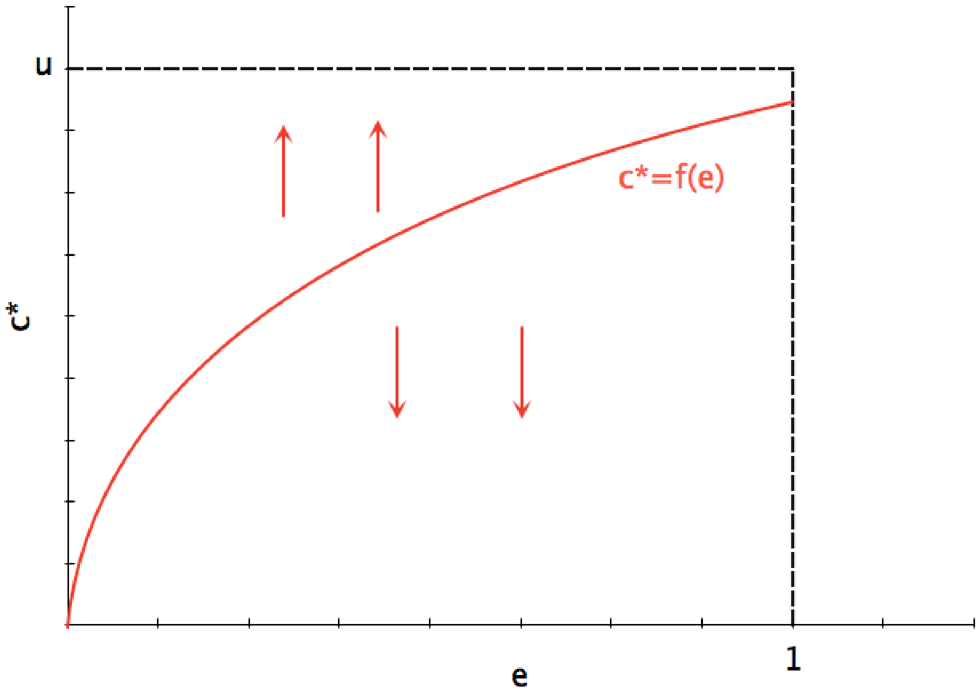
Does a steady state exist in the first place? As we will see, this is not exactly guaranteed [1]. Clearly, $\dot{c}(t) = \dot{e}(t) = 0$ generates two types of relationships between c^* and e. (SS1) and (SS2) define two locii of pairs (c, e) in the space of equilibrium paths: $\dot{c}(t) = 0 \Rightarrow c^* = f(e)$ and $\dot{e}(t) = 0 \Rightarrow e = h(c^*)$. A steady state only exists if both equations are simultaneously satisfied. That is, if the two locii cross. We proceed to analyse each of the locii separately.

1. $\dot{c}(t) = 0$. Recall that

$$c^* = \frac{b(e)u - a\int_{\underline{c}}^{c^*}(c^* - c)dG(c)}{r + b(e)} \tag{SS1}$$

Under the assumptions that $\underline{c} \geq 0$ and $b(0) = 0$, this curve passes through the origin. That is, $(0, 0) \in \{(e, c^*) : c^* = f(e)\}$. Propositions 3.2 and 3.3 also establish that, if $b(e)$ is continuous, increasing and concave, then this function will be continuous, increasing and concave. Furthermore, it is bounded above by u. Why? Exactly because it is never optimal to have $c^* \geq u$. Optimality of behaviour implies that $c^* < u, \forall e \in [0, 1]$. Furthermore, it can be shown that if $c^* > f(e)$, then $\dot{c}(t) > 0$ and, conversely, if $c^* < f(e)$ then $\dot{c}(t) < 0$ (this is in fact hinted by Proposition 3.2, but we shall not prove it here). The dynamics of this locus are presented in Figure 3.2.

Figure 3.2: The $c^* = f(e)$ locus



[1]In terms of *active* steady states, those where $e > 0$. The trivial inactive steady state always exists.

2. $\dot{e}(t) = 0$. We saw that

$$G(c^*) = \frac{eb(e)}{a(1-e)} \tag{SS2}$$

Defines the $e = g(c^*)$ relationship. Note first that while the origin belongs to this locus, there may be a discontinuity at $e = 0$. This is because $G(c) = 0$ for any $c < \underline{c}$. However, since $b(0) = 0$ and $b'(e) > 0$, having $G(c) = 0$ is only compatible with having $e = 0$. Thus there is a discontinuity point at 0 (if $\underline{c} > 0$), but the origin does belong to the locus. Once the origin is dealt with, the curve 'starts' at the point $(0, \underline{c})$. As $c^* \to \infty$, $G(c^*) \to 1$, and the expression becomes

$$a(1-\bar{e}) = \bar{e}b(\bar{e})$$

Thus, in the (e, c^*) space, the curve $e = h(c^*)$ will have a vertical asymptote at $\bar{e}$, which may or may not be 1. The shape of the curve can be inferred through some calculus. The first derivative of e with respect to c^* is obtained by differentiating the expression with respect to c^*:

$$ag(c^*)(1-e) - aG(c^*)\frac{\partial e}{\partial c^*} = b(e)\frac{\partial e}{\partial c^*} + eb'(e)\frac{\partial e}{\partial c^*}$$

Rearranging gives us

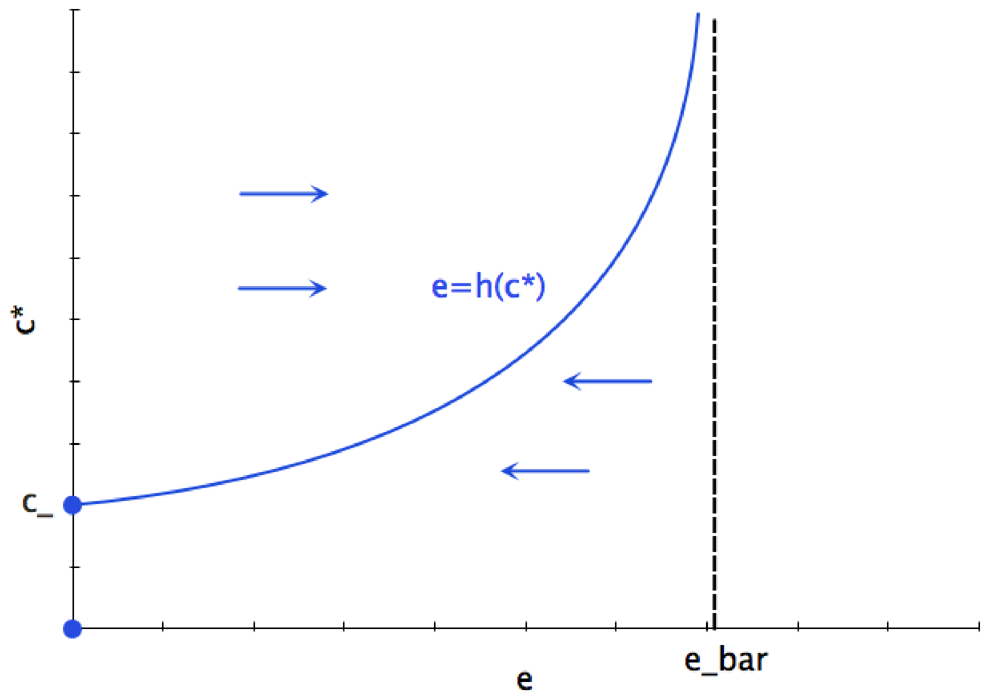
$$\frac{\partial e}{\partial c^*} = \frac{ag(c^*)(1-e)}{b(e) + eb'(e) + aG(c^*)}$$

so that having $b'(e) > 0$ is sufficient for the above derivative to be increasing. Regarding concavity, differentiate the expression again and rearrange to obtain

$$\frac{\partial^2 e}{\partial (c^*)^2} = \frac{ag'(c^*)(1-e) - 2ag(c^*)\frac{\partial e}{\partial c^*} - [2b'(e) + eb''(e)]\left(\frac{\partial e}{\partial c^*}\right)^2}{b(e) + aG(c^*) + eb'(e)}$$

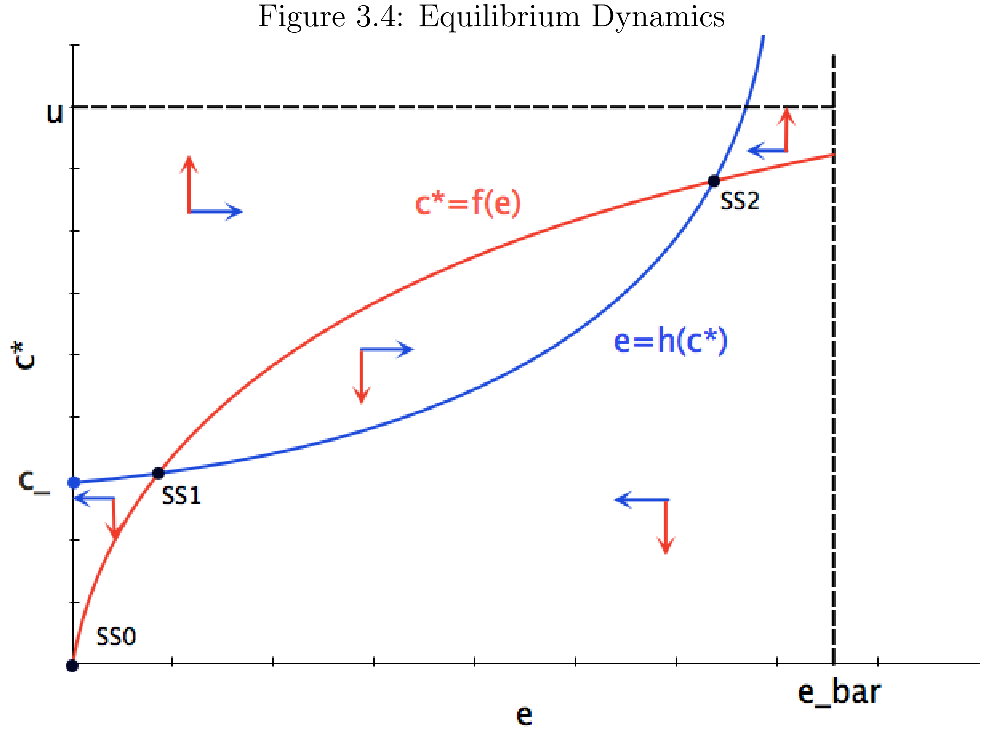
So that in general the sign of the derivative will depend on the sign of $g'(c^*)$ and $b''(e)$. For graphical analysis purposes, assume that the sign of the derivative is positive, so that $e = h(c^*)$ is shaped as a convex function. Once again, with some further restrictions we can show that $e > h(c^*) \Rightarrow \dot{e}(t) < 0$ and $e < h(c^*) \Rightarrow \dot{e}(t) > 0$, so that contrary to what happened with the $c^* = f(e)$ locus, this variable is *stable*, in the sense that its value tends to return to the stationary locus upon any shock/deviation. The dynamics are summarised by Figure 3.3.

Figure 3.3: The $c^* = f(e)$ locus



Putting everything together, and maintaining the $b'(e) > 0$ assumption gives us a possible scenario for the model dynamics:

Figure 3.4: Equilibrium Dynamics



In this case, there are three possible steady states: $(SS0, SS1, SS2)$: a degenerate steady state at the origin, with no production, a low activity steady state and a high activity steady state. Note that none of them may be stable, and we shall abstract of any stability properties (as the analysis would require much stronger assumptions than those we have made so far).

Welfare and Aggregate Demand Management

It is also interesting to note how crucial is the $b'(e) > 0$ assumption: if we had imposed, for example, that $b(e) = \bar{b}$ for $e > 0$ and $b(0) = 0$ then the $c^* = f(e)$ locus would be a horizontal line at $\bar{b}$ with a discontinuity at the origin. Therefore, only two steady states would remain: the origin and an intermediate steady state with an activity level between SS1 and SS2. This suggests that it is precisely the existence of the congestion externality that opens the possibility of what Diamond called **aggregate demand management** (whose defense was the whole point of the article): due to the existence of the externality, the market is subject to coordination failures. That is, the economy can become stuck on low activity steady states when high activity ones are available, due to the fact that individuals do not take the externality into consideration when deciding what reservation cost of climbing trees should they set. The higher the cost they choose, the more activity there will be on the exchange sector and the easier it becomes for them to trade. But individuals do not account for that, hence the economy may end up trapped in inefficient steady states.

Clearly, the First Welfare Theorem does not apply in this context, due to several reasons: first of all, the framework is not Walrasian - there is no such thing as a 'market' where prices are set (there are not even prices in this model). Secondly, there is a non-pecuniary externality, hence even if markets existed, they would not be complete.

It should be emphasised that no steady state may exist under certain distributional assumptions for G. Furthermore, even if steady states exist, as in Figure 4, none of them may be stable (excepting the non-active steady state at the origin). Thus it is possible for an economy to remain in equilibrium forever without ever converging to a steady state.

Welfare Analysis and the Planner's Problem

We conclude this section by undertaking some **Welfare Analysis**, motivated by the presence of the externality. Clearly, levels of activity and reservation costs chosen by a social planner will not coincide with those induced by a private equilibrium due to the fact that, as explained, agents do not account for the fact that b is a function of e. Define the instantaneous payoff for the planner as

$$U[c^*(t), e(t)] = e(t)b[e(t)]u - a[1 - e(t)] \int_{\underline{c}}^{c^*(t)} cdG(c)$$

Interpretation? At each instant, $e(t)b[e(t)]$ agents trade and earn utility u. At the same time, $a[1-e(t)]$ agents come across trees and incur in an expected cost $\hat{c}$, which is the expected value of c between $\underline{c}$ and the reservation cost $c^*(t)$. The planner seeks to maximise the present discounted value of social welfare subject to the law of motion for agents in the exchange sector. This suggests that the solution to this problem will, in fact, be a **Constrained Pareto Optimal** allocation: the planner chooses the level of activity and reservation cost subject to the constraint that such sequence $\{e(t), c^*(t)\}_{t\geq 0}$ must be supportable as a private equilibrium. The problem is thus

$$W_0 = \max_{\{e(t), c^*(t)\}_{t\geq 0}} \int_0^\infty e^{-rt} U[c^*(t), e(t)] dt$$

subject to

$$\begin{aligned}
\dot{e}(t) &= a[1-e(t)]G[c^*(t)] - e(t)b[e(t)] \\
e(t) &\in [0,1] \\
e(0) &= e_0 \text{ given}
\end{aligned}$$

Before attacking this problem, note the following:

Remark 3.4.2. *It turns out that* any *steady state that arises as a private equilibrium is plagued by too little activity. An alternative way to undertake welfare analysis is too look only at what happens on the steady state. Let $\bar{W}$ denote the present discounted value of social welfare at the steady state. We can write the objective function of the planner evaluated at a steady state (c^*, e) as*

$$\bar{W} = \int_0^\infty e^{-rt} U(c^*, e) dt = \frac{U(c^*, e)}{r}$$

where (c^*, e) solve (SS1), (SS2). Diamond shows that, as long as the equilibrium is interior (i.e. active, $e > 0$), we generically have

$$\frac{\partial \bar{W}}{\partial c^*} > 0$$

Thus no steady state arising as a private equilibrium is efficient, as there are social welfare gains from making agents climb higher trees and trade more. The drawback of this type of analysis is that by focusing on the steady state only, it completely ignores transitional dynamics. Hence it should be seen as a minimal test for necessity of policy intervention.

Back to the dynamic problem, the (present value) Hamiltonian for the planner's problem is the following

$$\mathcal{H} = e^{-rt} U[c^*(t), e(t)] + \mu(t)\{a[1-e(t)]G[c^*(t)] - e(t)b[e(t)]\}$$

The planner chooses the control $c^*(t)$, while $e(t)$ is a state variable [2]. $\mu(t)$, the multiplier, is the costate variable and represents the value of an additional agent on the exchange

[2]It is relevant to emphasise that $e(t)$ is an endogenous state, hence it is listed as a control variable for the planner in the original statement of the problem.

sector along an optimal path for the planner. The FOC are

$$\frac{\partial \mathcal{H}}{\partial c} = 0$$
$$\frac{\partial \mathcal{H}}{\partial e} = -\dot{\mu}$$

That is

$$-e^{-rt}a[1-e(t)]c^*(t)g[c^*(t)] + \mu(t)a[1-e(t)]g[c^*(t)] = 0$$
$$e^{-rt}\left\{b[e(t)]u + e(t)b'[e(t)]u + a\int_{\underline{c}}^{c^*(t)} cdG(c)\right\} - \mu(t)\left\{aG[c^*(t)] + b[e(t)] + e(t)b'[e(t)]\right\} = -\dot{\mu}(t)$$

This looks rather messy, especially because both conditions directly depend on the value and rate of change of a variable that does not interest us directly - the costate. Thus we aim to eliminate that variable from the problem. The first FOC can be simplified so as to obtain

$$e^{-rt}c^*(t) = \mu(t)$$

It must hold for all t, so that taking logs

$$-rt + \log c^*(t) = \log \mu(t)$$

and differentiating with respect to t

$$-r + \frac{\dot{c}^*(t)}{c^*(t)} = \frac{\dot{\mu}(t)}{\mu(t)}$$

Now, divide both sides of the second FOC by $\mu(t)$ to get

$$\frac{e^{-rt}}{\mu(t)}\left\{b[e(t)]u + e(t)b'[e(t)]u + a\int_{\underline{c}}^{c^*(t)} cdG(c)\right\} - \left\{aG[c^*(t)] + b[e(t)] + e(t)b'[e(t)]\right\} = -\frac{\dot{\mu}(t)}{\mu(t)}$$

Notice that we can replace the LHS by what we have found by manipulating the first FOC. Furthermore, from its original form, we know that

$$\frac{e^{-rt}}{\mu(t)} = \frac{1}{c^*(t)}$$

So that replacing both this term and the LHS leaves us with a condition that no longer depends on the costate

$$\frac{1}{c^*(t)}\left\{b[e(t)]u + e(t)b'[e(t)]u + a\int_{\underline{c}}^{c^*(t)} cdG(c)\right\} - \left\{aG[c^*(t)] + b[e(t)] + e(t)b'[e(t)]\right\} = r - \frac{\dot{c}^*(t)}{c^*(t)}$$

or, rearranging and simplifying

$$\dot{c}^*(t) = rc^*(t) - [u - c^*(t)]\{b[e(t)] + e(t)b'[e(t)]\} + a\int_{\underline{c}}^{c^*(t)} [c^*(t) - c]dG(c) \qquad \text{(PP)}$$

(PP) along with the law of motion of $e(t)$ define the solution to the planner's problem. Recall that in the private equilibrium, the equilibrium conditions were, precisely, the law of motion for $e(t)$ and we had

$$\dot{c}^*(t) = rc^*(t) - b[e(t)][u - c^*(t)] + a \int_{\underline{c}}^{c^*(t)} [c^*(t) - c]dG(c) \tag{E1}$$

Clearly, these two conditions look strikingly similar. What is the only difference? Precisely the $b'[e(t)]$ term, which enters (PP) but not (E1)! They only coincide when $b'[e(t)] = 0$, **when there is no externality**. In terms of dynamics, the planner solution generates a $c^* = f(e)$ locus that is *above* the one generated by the private equilibrium. Hence the high activity steady state will feature even more activity.

What can policymakers do to correct this market failure? Some potential policies are:

1. Subsidising production, giving a subsidy to each agent who climbs a tree.
2. Imposing a Pigou tax on agents who do not climb trees.
3. Subsidise trade.
4. etc.

3.4.2 Mortensen & Pissarides 1994 - Job Creation and Job Destruction in the Theory of Unemployment

M-P take Diamond's coconut model and apply it to a labour search setting. As mentioned in the introduction to this section, this model will endogenise not only the rate of arrival of job offers but also the wage that is bundled with each of these offers.

The argument put forward by the authors is that congestion externalities, in the same spirit of those in the Diamond model, are extremely common in the labour market. If more firms are looking for workers, it becomes easier to find a job, while the reverse is true when more unemployed workers look for jobs.

Environment

The environment is as follows:

1. There is a continuum of workers of measure one, with identical preferences. Each worker can be employed or unemployed. Let u denote the measure of unemployed workers.

2. There is also a continuum of firms, measure one. Each firm can hire one worker. Firms can be vacant or filled, let v denote the measure of vacant firms.

3. Firms and workers interact on a frictional labour market. Meetings between workers and firms are ruled by a constant returns to scale **matching function** $m(u, v)$. The matching function satisfies $m_1, m_2 > 0$ and $m_{11}, m_{22} < 0$.

4. Time is continuous and agents are infinitely lived.

5. Matching is random and takes place only between unemployed workers and vacant firms. That is, employed workers and filled firms do not search/are not matched.

6. Production takes place when a firm and a worker are matched. Let p denote the output from a match.

7. Let λ be the Poisson rate of a productivity/destruction shock that affects matches. With arrival rate λ, a shock hits the match between worker and firm and makes productivity become equal to zero forever.

For a vacant firm, the probability of finding a worker is equal to the total number of matches divided by the number of firms that are looking for workers, hence $\frac{m(u,v)}{v}$. Since m is CRS, hence homogeneous of degree one in both arguments, we can write

$$\Pr[\text{finding a worker}] = \frac{m(u, v)}{v} = m\left(\frac{u}{v}, 1\right)$$

Let $\theta := \frac{v}{u}$, this then allows us to write

$$m\left(\frac{u}{v}, 1\right) = m\left(\frac{1}{\theta}, 1\right) =: q(\theta)$$

That is, the probability of a vacant firm finding a worker does not depend on the unemployment/vacancy levels, but rather on their ratio θ due to the CRS structure we have imposed.

For a worker looking for a vacant firm, the probability of coming across one is equal to $\frac{m(u,v)}{u}$. Once again, we can take advantage of CRS to write

$$\frac{m(u, v)}{u} = m\left(1, \frac{v}{u}\right) = m(1, \theta) = \theta m\left(\frac{1}{\theta}, 1\right) = \theta q(\theta)$$

Thus the **contact rates** for firms and workers are $q(\theta)$ and $\theta q(\theta)$, respectively. θ, the ratio of vacancies to unemployed workers can be read as a measure of **market tightness** (i.e., how easy it is to find a job or not). We therefore increment the environment with the following assumption:

Assumption 3.4.3. *Unemployed workers contact vacant firms at Poisson rate $\theta q(\theta)$. Vacant firms contact unemployed workers at Poisson rate $q(\theta)$.*

Remark 3.4.3. *At this stage, we already have enough information to conduct some analysis on how the contact rates respond to changes in market tightness. Note that*

$$q'(\theta) = -\frac{1}{\theta^2} m_1 \left(\frac{1}{\theta}, 1\right) < 0$$

Since $m_1 > 0$. Thus an increase in the vacancy/unemployment ratio decreases the rate at which firms contact workers. This is rather intuitive: $\theta \uparrow$ means that either there are less workers looking for firms or there are more firms looking for workers. Hence it is more difficult for an individual firm to come across a worker. We also have that

$$\frac{d}{d\theta}[\theta q(\theta)] = q(\theta) - \theta q'(\theta) = m\left(\frac{1}{\theta}, 1\right) - \frac{1}{\theta} m_1 \left(\frac{1}{\theta}, 1\right)$$

In principle, this sign is ambiguous. We can, however, take advantage of the fact that m is homogeneous of degree one to apply Euler's Theorem for homogeneous functions and write

$$m\left(\frac{1}{\theta}, 1\right) = \frac{1}{\theta} m_1 \left(\frac{1}{\theta}, 1\right) + m_2 \left(\frac{1}{\theta}, 1\right)$$

Thus

$$\frac{d}{d\theta}[\theta q(\theta)] = \frac{1}{\theta} m_1 \left(\frac{1}{\theta}, 1\right) + m_2 \left(\frac{1}{\theta}, 1\right) - \frac{1}{\theta} m_1 \left(\frac{1}{\theta}, 1\right) = m_2 \left(\frac{1}{\theta}, 1\right) > 0$$

Since $m_2 > 0$ by assumption. Thus the effect of less unemployment/more vacancies on the contact rate for workers is the opposite, as we would expect. These effects are analogous to the $b'(e) > 0$ assumption in the Diamond Model, with the caveat that they now have opposite signs to different types of agents in the market (workers and firms).

Unemployment and the Beveridge Curve

The productivity/destruction shock aims at introducing turnover in the model and is a proxy for job destruction in the economy. This means that, given the current knowledge of the model, we can already derive the law of motion of unemployment. At each instant, $(1-u)$ workers are employed and are subject to the destruction shock with rate λ, hence $\lambda(1-u)$ workers become unemployed. At the same time, u workers are unemployed and contact vacant firms at rate $\theta q(\theta)$, thus

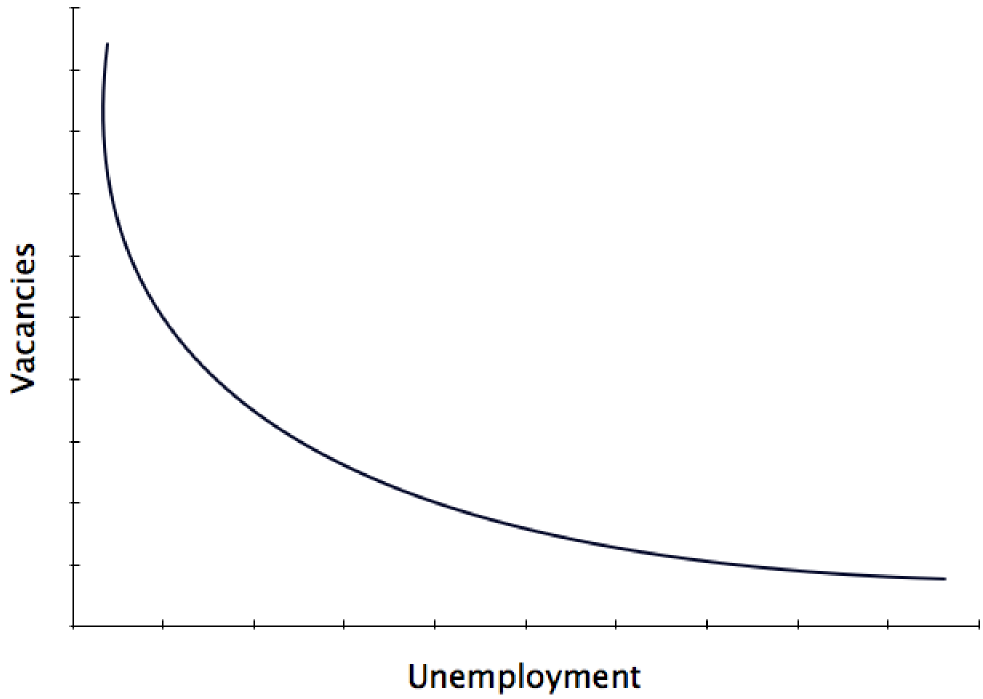
$$\dot{u} = \lambda(1-u) - \theta q(\theta) u$$

At this point, we are assuming that any firm hit by a destruction shock *fires* the worker and any worker who contacts a firm *accepts* the job offer. These assumptions will be justified shortly. Note that in the steady state, when $\dot{u}$, a relationship between v and u arises through θ:

$$u = \frac{\lambda}{\lambda + \theta q(\theta)}$$

Since $\theta q(\theta)$ is a positive function of θ, it is a positive function of v. Hence the above expression defines a negative relationship between u and v - this is the so-called **Beveridge Curve**.

Figure 3.5: Beveridge Curve



Value Functions of the Firm

We proceed by deriving the **value functions** for this model. Note that now we have two types of agents, and each of them can be in one of two states. This means that we will have a total of *four* different value functions to analyse. We start by looking at firms.

These can either be filled or vacant. Assume that, while searching for workers, firms must pay a *search cost* equal to $pc > 0$. There is a large number of firms that can choose to enter the market (there is *free entry*). This means that, in equilibrium, we will have the value of vacancies equal to zero: if positive, firms would keep entering the market until that value comes down to zero, and if negative firms would leave the market. These dynamics will become clearer in a moment.

Let V denote the value of a vacant firm and J the value of a firm who has filled a vacancy. Once again, we start by discretising the problem and then letting the time interval go to zero. Assuming that firms share the same discount factor as workers, $\beta(\Delta) = \frac{1}{1+r\Delta}$, a firm with an open vacancy pays $cp\Delta$ per time period. With probability $q(\theta)\Delta$, it contacts a worker and fills the vacancy, earning the continuation value $J(t+\Delta)$. With probability $1 - q(\theta)\Delta$, it fails to contact a worker and continues with $V(t + \Delta)$. Thus

$$V(t) = \frac{1}{1 + r\Delta}[-cp\Delta + q(\theta)\Delta J(t + \Delta) + [1 - q(\theta)\Delta]V(t + \Delta) + o(\Delta)]$$

Rearranging and dividing both sides by Δ yields

$$rV(t) = -cp + q(\theta)[J(t+\Delta) - V(t+\Delta)] + \frac{V(t+\Delta) - V(t)}{\Delta} + \frac{o(\Delta)}{\Delta}$$

So that taking limits to zero

$$rV(t) = -cp + q(\theta)[J(t) - V(t)] + \dot{V}(t)$$

For a firm which has filled a vacancy, it earns p and pays some wage w per time period Δ. With probability $\lambda\Delta$, it is affected by a negative productivity shock and fires the worker, earning $V(t+\Delta)$. With probability $1-\lambda\Delta$, it remains filled and earns $J(t+\Delta)$. The value is therefore

$$J(t) = \frac{1}{1+r\Delta}[(p-w)\Delta + \lambda\Delta V(t+\Delta) + (1-\lambda\Delta)J(t+\Delta) + o(\Delta)]$$

Rearranging and dividing by Δ

$$rJ(t) = p - w + \lambda[V(t+\Delta) - J(t+\Delta)] + \frac{J(t+\Delta) - J(t)}{\Delta} + \frac{o(\Delta)}{\Delta}$$

Taking limits

$$rJ(t) = p - w + \lambda[V(t) - J(t)] + \dot{J}(t)$$

We will be focusing on stationary environments (steady states), hence will set $\dot{V}(t) = \dot{J}(t) = 0$. Note that, in this case, free entry requires that $V(t) = 0, \forall t \geq 0$ (for simplicity, let us omit the time argument from the value functions). From the value function for an open vacancy, this implies

$$V = 0 \Leftrightarrow J = \frac{cp}{q(\theta)}$$

That is, in equilibrium, the value of a filled vacancy should equal the (instantaneous) cost of searching for a match divided by the rate of matching, or the *expected hiring cost* (on average, how much does it cost to look for and hire a worker). Similarly, imposing $V = 0$ on the expression for J leaves us with

$$J = \frac{p-w}{r+\lambda}$$

which is the natural condition stating that the value of a filled vacancy is equal to the expected present value of profits that are extracted from filling a vacancy. Note that, similarly to what happened when we allowed for workers to 'die' in the context of one-sided job search, the rate of job destruction acts as an incremental discount factor. Thus, in equilibrium:

$$\frac{cp}{q(\theta} = \frac{p-w}{r+\lambda} \Rightarrow p = w + (r+\lambda)\frac{pc}{q(\theta)}$$

This is the so-called **Job Creation Condition**: it states that the marginal productivity of labour p should equal the wage plus a wedge term. In Walrasian terms, this additional term is equal to zero, and we obtain the usual condition

$$MP_L = MC_L$$

However, in this case we will generally observe $p > w$ (the wedge is positive). This is due to the existence of frictions in the labour market: it is costly for firms to hire workers. Therefore, they extract some additional surplus from workers to compensate for that. If $c = 0$, the wedge would disappear. It is also interesting to note that the wedge also disappears when $q(\theta) \to \infty$, which would happen when $\theta \to 0$ (if we impose a Inada condition on m). That is, when there is a very small number of vacancies or a very large number of workers looking for a job, the firm is able to fill a vacancy almost instantly. Therefore, it does not incur in high search costs and is willing to hire a worker at a lower wedge (thus transferring surplus to the worker). It is this sort of dynamics that motivate the condition $V = 0$.

Value Functions of Workers

Letting z denote unemployment benefits, the derivation is standard and we skip most of it. Letting W denote the value of employment and U the value of unemployment, we obtain, imposing stationarity already

$$\begin{aligned} rU &= z + \theta q(\theta)(W - U) \\ rW &= w + \lambda(U - W) \end{aligned}$$

Is there anything fishy in the above value functions? Note that, without further assumptions, we would usually get, for unemployment, something along the lines of

$$rU = z + \theta q(\theta)\mathbb{E}_w \max\{W - U, 0\}$$

Thus we are implicitly assuming that the worker *always* accepts a job offer. Once again, this can be (at least partially) justified with the model's assumptions.

Claim 3.4.1. *A worker always accepts a job offer.*

Proof. Rewrite the above value functions as

$$\begin{aligned} [r + \theta q(\theta)]U &= z + \theta q(\theta)W \\ (r + \lambda)W &= w + \lambda U \end{aligned}$$

Plug each one in the other to obtain

$$\begin{aligned} rU &= \frac{(r + \lambda)z + \theta q(\theta)w}{r + \lambda + \theta q(\theta)} \\ rW &= \frac{\lambda z + [r + \theta q(\theta)]w}{r + \lambda + \theta q(\theta)} \end{aligned}$$

Subtracting one from the other

$$r(W - U) = \frac{r(w - z)}{r + \lambda + \theta q(\theta)}$$

So that as long as $w \geq z$, we will have $W \geq U$. z is the reservation wage and, as we will see in the next section, w is constructed such that $w \geq z$ always holds in equilibrium. □

Wage Determination

In the Diamond model, all agents were the same. Therefore, whenever agents were matched and traded, each one would eat a coconut, period. Now, however, things are not that simple. The instantaneous value of a match is equal to $p - w$. This is a pie that ought to be shared among worker and firm.

The model assumes that the wage is the outcome of a **Nash Bargaining Process**. Let $\beta \in [0, 1]$ denote the bargaining weight for the worker and $1 - \beta$ the bargaining weight for the firm. Then, the wage solves the Generalised Nash Problem

$$\max_{w} [W - U]^{\beta} [J - V]^{1-\beta}$$

Note that (U, V) is the threat point - neither the worker is willing to obtain a wage that induces $W < U$ nor will the firm concede to a wage that leaves it with $J < V$ - the worker has the option of remaining unemployed, as the firm has the option to remain vacant. Taking into account that only W and J, not the threat points, depend on w, the FOC will be

$$\beta(W - U)^{\beta-1}(J - U)^{1-\beta}\frac{\partial W}{\partial w} + (1 - \beta)(J - V)^{-\beta}(W - V)^{\beta}\frac{\partial V}{\partial w} = 0$$

We know that

$$J = \frac{p - w}{r + \lambda} \Rightarrow \frac{\partial V}{\partial w} = \frac{-1}{r + \lambda}$$
$$W = \frac{w + \lambda U}{r + \lambda} \Rightarrow \frac{\partial W}{\partial w} = \frac{1}{r + \lambda}$$

hence $\frac{\partial W}{\partial w} = -\frac{\partial J}{\partial w}$, so that the FOC becomes

$$\beta(W - U)^{\beta-1}(J - V)^{1-\beta} = (1 - \beta)(J - V)^{-\beta}(W - U)^{\beta}$$

or more simply

$$\frac{\beta}{W - U} = \frac{1 - \beta}{J - V}$$

which, rearranging once more, leaves us with a very intuitive condition

$$W - U = \beta[W - U + J - V]$$

The LHS is the surplus of the worker, and this condition tells us that the optimal outcome is to set it equal to a fraction β of the total surplus, obtained by adding the surplus of the worker and that of the firm. Or

$$W = U + \beta[W - U + J - V]$$

the total gain from the process is equal to the reservation utility (the outside option) plus such a fraction of the total net surplus from the process.

We would appreciate eliminating the value functions from this problem. For the firm, this is very easy, given the no free entry condition

$$J - V = \frac{p - w}{r + \lambda} - 0 = \frac{p - w}{r + \lambda}$$

whereas for the worker, as we have seen

$$W - U = \frac{w + \lambda U}{r + \lambda} - U = \frac{w - rU}{r + \lambda}$$

Thus the FOC becomes

$$\beta(p - w) = (1 - \beta)(w - rU)$$

yielding

$$w = \beta p + (1 - \beta) rU$$

Thus the optimal wage is a weighted average of the total productivity generated by the worker and the reservation payoff, which is equal to the flow value of unemployment, rU. We can further eliminate this term by adopting an extremely clever method. Recall that, from the original value function for unemployment

$$rU = z + \theta q(\theta)(W - U)$$

From the FOC for the Nash Problem, $W - U = \frac{\beta}{1-\beta}(J - V)$ which, by the free entry condition, could also be written as $\frac{\beta}{1-\beta}\frac{pc}{q(\theta)}$. Replacing in the expression:

$$rU = z + \frac{\beta}{1 - \beta}\theta pc$$

So that, finally replacing in the wage equation gives us the final expression for this variable

$$w = \beta p(1 + \theta c) + (1 - \beta)z$$

This gives us the wage as a function of a sole endogenous variable: θ.

Remark 3.4.4. *Let $p > z$ (not an unreasonable assumption). Then, it automatically follows that $w \geq z$, strictly for $\beta > 0$.*

Equilibrium (Steady State)

We are now ready to study the equilibrium for this model. For simplicity, and this should once again be emphasised, we are only focusing on steady states, where value functions are stationary.

Definition 3.4.3. *A* ***search equilibrium*** *(stationary) is a triple (u, θ, w) that satisfies the following conditions*

$$u = \frac{\lambda}{\lambda + \theta q(\theta)} \tag{SSU}$$

$$p - w = (r + \lambda)\frac{pc}{q(\theta)} \tag{JC}$$

$$uw = \beta p(1 + \theta c) + (1 - \beta)z \tag{WC}$$

Note that if we know market tightness θ, we immediately obtain u from (SSU) and w from (WC), as these two variables are determined separately. Therefore, if we determine θ, we have the entire equilibrium characterised.

Combine (JC) and (WC) to obtain, by substituting out w:

$$(1-\beta)(p-z) - \frac{r+\lambda+\beta\theta q(\theta)}{q(\theta)} pc = 0 \Leftrightarrow T(\theta) = 0$$

That is, an expression which depends only on parameters and that pins down θ. Solving this equation gives us θ, which then allows for determination of (u, w).

Claim 3.4.2. $T'(\theta) < 0$

Proof.

$$T'(\theta) = -pc\frac{\beta q(\theta)^2 - q'(\theta)(r+\lambda)}{q(\theta)^2} < 0$$

Follows from $q'(\theta) < 0$ $\square$

A sufficient condition for the equilibrium to exist is then for $T(\theta) = 0$ to be satisfied for some $\theta \geq 0$. This will usually be true under the assumption that $p > z$.

Comparative Statics

Table 3.1 summarises some of the most relevant effects of parameters on the endogenous variables (θ, u, w, v)

Table 3.1: Some Comparative Statics

	p	z	β	λ	r
θ	$+$	$-$	$-$	$-$	$-$
u	$-$	$+$	$+$	$+$	$+$
w	$+$	$+$	$+$	$-$	$-$
v	$+$	$-$	$-$	?	$-$

1. $p \uparrow$ is an increase in the marginal productivity of a worker. Naturally, this benefits both workers and firms: wages increase and unemployment goes down as more workers accept offers. There is a somehow artificial effect on the number of vacancies, since the cost of search is scaled by p (a helpful normalisation), hence the value of a filled vacancy has now increased, prompting more firms to enter the market looking for workers. As $v \uparrow, u \downarrow$, we observe an increase in θ.

2. $z \uparrow$, an increase in unemployment benefits raises the outside option of the worker. This leads to an increase in wages as workers are now more 'protected', and unemployment rises as less workers decide to accept jobs. Also, the loss of margin by firms may drive some firms out of the market, thereby reducing vacancies. As $v \downarrow, u \uparrow$ we observe $\theta \downarrow$.

3. $\beta \uparrow$, an increase in the bargaining power for workers has exactly the same effects as an increase in z. The logic is mostly the same: workers have more power and firms have less.

4. $\lambda \uparrow$, an increase in the rate of job destruction leads, as we would expect, to a higher unemployment rate. Furthermore, workers lose bargaining power and wages decrease. This happens because, as we have seen, λ increments the rate at which firms discount the value of a filled vacancy. So, basically, firms now value filled positions less. It can be shown that market tightness decreases, $\theta \downarrow$. Given that this effect coexists with $u \uparrow$, the effect on the total level of vacancies is ambiguous.

5. $r \uparrow$, an increase in the discount rate has similar effects to the increase in λ. However, it can be unambiguously shown that vacancies decrease in this case.

Planner's Problem

How does the congestion externality affect the welfare properties of this economy? As we have seen, the entire equilibrium can be characterised by the variable θ. Let us consider the case of a planner who can choose a path of socially optimal levels of market tightness $\theta(t)$. Once again, we are dealing with *Constrained Pareto Optimality*, as the planner is subject to the physical constraints of the economy: the equilibrium law of motion of unemployment. Consider, for simplicity, an utilitarian planner who places the same weight on workers and firms.

We start by specifying the instantaneous social welfare function. At each instant, social benefits are equal to p for each employed worker, in measure $1-u$ and equal to z for each unemployed worker in measure u. At the same time, a measure $v = \theta u$ of firms are vacant and incur in search costs pc. Therefore, the social return function is given by

$$U[u(t), \theta(t)] = [1 - u(t)]p + u(t)z - pc\theta(t)u(t)$$

Consider, for simplicity, that the planner's discount factor is given by e^{-rt}. Then, the problem can be constructed as that of a planner who chooses $\theta(t)$ subject to $\dot{u}(t)$

$$\max_{\{\theta(t)\}_{t\geq 0}} \int_0^\infty e^{-rt}\{[1 - u(t)]p + u(t)z - pc\theta(t)u(t)\}$$

subject to

$$\dot{u}(t) = \lambda[1 - u(t)] - \theta(t)q[\theta(t)]u(t)$$

The present-value Hamiltonian for the planner is then (omitting, for simplicity, the dependence of variables on t):

$$\mathcal{H} = e^{-rt}[(1-u)p + uz - pc\theta u] + \mu[\lambda(1-u) - \theta q(\theta)u]$$

The FOC are

$$\frac{\partial \mathcal{H}}{\partial \theta} = 0 \Leftrightarrow -e^{-rt}pcu - \mu u[q(\theta) + \theta q'(\theta)] = 0$$
$$\frac{\partial \mathcal{H}}{\partial u} = -\dot{\mu} \Leftrightarrow e^{-rt}[-p + z - pc\theta] - \mu[\lambda + \theta q(\theta)] = -\dot{\mu}$$

Note that the first FOC can be rewritten as

$$e^{-rt}pc + \mu q(\theta)\left[1 + \frac{\theta q'(\theta)}{q(\theta)}\right] = 0$$

Let $\eta(\theta) := 1 + \frac{\theta q'(\theta)}{q(\theta)}$. Then, we can rewrite it as

$$-\mu = \frac{e^{-rt}pc}{q(\theta)\eta(\theta)}$$

We use the same trick as in the Diamond model to eliminate the costate from the problem: given that the first FOC must hold for all t, differentiate it with respect to t to obtain

$$-\dot{\mu} = \frac{-re^{-rt}pcq(\theta)\eta(\theta) - e^{-rt}pc[q'(\theta)\eta(\theta) + \eta'(\theta)q(\theta)]\dot{\theta}}{[q(\theta)\eta(\theta)]^2} = -r\frac{e^{-rt}pc}{q(\theta)\eta(\theta)} - \frac{e^{-rt}pc[q'(\theta)\eta(\theta) + \eta'(\theta)q(\theta)]}{[q(\theta)\eta(\theta)]^2}\dot{\theta}$$

Notice that the first term on the RHS is equal to $r\mu$ from the original FOC. Similarly, we can also extract $-\mu$ from the first term and write it as

$$\frac{e^{-rt}pc[q'(\theta)\eta(\theta) + \eta'(\theta)q(\theta)]}{[q(\theta)\eta(\theta)]^2}\dot{\theta} = -\mu\frac{[q'(\theta)\eta(\theta) + \eta'(\theta)q(\theta)]}{q(\theta)\eta(\theta)}\dot{\theta} = -\mu\left[\frac{q'(\theta)}{q(\theta)} + \frac{\eta'(\theta)}{\eta(\theta)}\right]\dot{\theta}$$

Dividing and multiplying the second term by θ, our original expression then becomes

$$-\dot{\mu} = r\mu + \mu\left[\frac{q'(\theta)\theta}{q(\theta)} + \frac{\eta'(\theta)\theta}{\eta(\theta)}\right]\frac{\dot{\theta}}{\theta}$$

Now, recall that from the way we have defined $\eta(\theta)$:

$$\eta(\theta) = 1 + \frac{q'(\theta)\theta}{q(\theta)} \Rightarrow \frac{q'(\theta)\theta}{q(\theta)} = \eta(\theta) - 1$$

So that, replacing in the expression, and dividing the whole expression by μ leaves us with

$$-\frac{\dot{\mu}}{\mu} = r + \left[\frac{\eta'(\theta)\theta}{\eta(\theta)} - (1 - \eta(\theta))\right]\frac{\dot{\theta}}{\theta}$$

Thus we have written the growth rate of the costate as a function of parameters and θ. This will be useful to handle the second FOC. Take the second FOC and divide it throughout by μ

$$\frac{e^{-rt}}{\mu}[-p + z - pc\theta] - [\lambda + \theta q(\theta)] = -\frac{\dot{\mu}}{\mu}$$

From the first FOC, we know that $\frac{e^{-rt}}{\mu} = -\frac{q(\theta)\eta(\theta)}{pc}$, replacing this and replacing for $\frac{\dot{\mu}}{\mu}$ leaves us then with

$$-\frac{q(\theta)\eta(\theta)}{pc}[-p + z - pc\theta] - [\lambda + \theta q(\theta)] = r + \left[\frac{\eta'(\theta)\theta}{\eta(\theta)} - (1 - \eta(\theta))\right]\frac{\dot{\theta}}{\theta}$$

which no longer depends on the costate. Rearranging gives us then two conditions that describe the planner's problem: the one we have just found and the law of motion for unemployment, to which the planner is constrained

$$\frac{q(\theta)\eta(\theta)[p(1 + \theta c) - z]}{pc} - \lambda - \theta q(\theta) = r + \left[\frac{\eta'(\theta)\theta}{\eta(\theta)} - (1 - \eta(\theta))\right]\frac{\dot{\theta}}{\theta} \tag{SP1}$$

$$\dot{u} = \lambda(1 - u) - \theta q(\theta)u \tag{SP2}$$

An obvious remark is that the planner's decision is unaffected by the parameter β, that heavily influences the private equilibrium: the planner simply does not care how the surplus from a match is split between worker and firm.

Once again, and just like how we did with the private equilibrium, let us focus on the steady state. Imposing $\dot{\theta}) = \dot{u} = 0$, these conditions become

$$\frac{q(\theta)\eta(\theta)[p(1 + \theta c) - z]}{pc} - r - \lambda - \theta q(\theta) = 0 \tag{SSP1}$$

$$0 = \lambda(1 - u) - \theta q(\theta)u \tag{SSP2}$$

Recall that the steady state equilibrium conditions in the private case were

$$(1 - \beta)(p - z) - \frac{r + \lambda + \beta\theta q(\theta)}{q(\theta)}pc = 0 \tag{SSE1}$$

$$0 = \lambda(1 - u) - \theta q(\theta)u \tag{SSE2}$$

Naturally, (SSP2)=(SSE2). Now, rearrange (SSE1) to write it as

$$[1 - \beta(1 + \theta c)]p - (1 - \beta)z = \frac{(r + \lambda)pc}{q(\theta)}$$

We can also rearrange (SSP1) so that it becomes more reminiscent of the above condition

$$[\eta(\theta)(1 + \theta c) - c\theta]p - \eta(\theta)z = \frac{(r + \lambda)pc}{q(\theta)}$$

Notice that the RHS of (SSE1) and (SSP1) now coincide. Therefore, we would like to study conditions under which the LHS also coincides and, therefore, the steady state equilibrium coincides with the steady state chosen by the social planner. Clearly, this will happen whenever

$$\begin{aligned} \eta(\theta) &= 1-\beta \\ \Rightarrow 1+\frac{q'(\theta)\theta}{q(\theta)} &= 1-\beta \\ \Rightarrow \beta &= -\frac{q'(\theta)\theta}{q(\theta)} \end{aligned}$$

This is the so-called **Hosios Condition**: the private equilibrium will be efficient whenever the above condition holds.

Claim 3.4.3. $-\frac{q'(\theta)\theta}{q(\theta)}$ *is the elasticity of the matching function with respect to unemployment.*

Proof. Recall that

$$q'(\theta) = -\frac{1}{\theta^2} m_1\left(\frac{1}{\theta}, 1\right)$$

The elasticity of the matching function with respect to unemployment can be computed as

$$\begin{aligned} \frac{\partial m(u,v)}{\partial u}\frac{u}{m(u,v)} &= m_1(u,v)\frac{1}{m(u,v)/u} \\ &= m_1\left(\frac{1}{\theta}, 1\right)\frac{1}{m(1,\theta)} && m_1 \text{ homog. of deg. 0 and } m \text{ homog. of deg. 1} \\ &= m_1\left(\frac{1}{\theta}, 1\right)\frac{1}{\theta}\frac{1}{m\left(\frac{1}{\theta},1\right)} \\ &= -\theta m_1\left(\frac{1}{\theta}, 1\right)\left(-\frac{1}{\theta^2}\right)\frac{1}{m\left(\frac{1}{\theta},1\right)} \\ &= -\frac{\theta q'(\theta)}{q(\theta)} \end{aligned}$$

□

Thus the Hosios Condition tells us that the equilibrium will be efficient if the worker's bargaining power coincides with the elasticity of matching with respect to unemployment. Given that we have already suggested that the matching function is analogous to a production function, this condition is heavily reminiscent of the neoclassical efficiency rule that 'each factor of production should be paid a share of income equal to its relative contribution to production'. If matching is extremely sensitive to changes in unemployment, then this means that the workers' status can heavily affect the total surplus that is generated in the economy. Therefore, they should, in an efficient world, earn a greater share of income to account for their contribution to matching.

Problem? Due to the congestion externality, workers do not take into account that they influence the number of matches when optimising. The same happens with firms: they do not account for the fact that the total number of vacancies affects the number of matches that occur in each instant. Therefore, we have, once again, the congestion externality affecting the efficiency of this search economy.

The novelty here is that, due to the mechanism through which prices (wages) are set, there is a rough hope that the equilibrium may turn out to be efficient. However, note that this requires β, a parameter, to coincide with an elasticity that depends on the main endogenous variable θ. Therefore, it is safe to say that, generically, these economies will not be efficient - efficiency is only achieved for a measure zero of values for the parameters.

Therefore, **the equilibrium is generically inefficient due to the congestion externality**. As the worker starts looking for a job, he does not realise that unemployment goes up and everyone's chances of finding a job go down.

Dynamics of the Optimal Path

In order to study the dynamics of this model, it is useful to assume an explicit functional form for the matching function. Assume that it is of Cobb-Douglas type

$$m(u, v) = u^{\varepsilon} v^{1-\varepsilon}$$

This means that

$$q(\theta) = m\left(\frac{1}{\theta}, 1\right) = \theta^{-\varepsilon}$$

and

$$\theta q(\theta) = \theta^{1-\varepsilon}$$

So that

$$\begin{aligned} q'(\theta) &= -\varepsilon \theta^{-\varepsilon - 1} \\ \theta q'(\theta) &= -\varepsilon \theta^{-\varepsilon} \end{aligned}$$

Recall that the planner's optimality conditions, before imposing steady state, were

$$\frac{q(\theta)\eta(\theta)[p(1+\theta c) - z]}{pc} - \lambda - \theta q(\theta) = r + \left[\frac{\eta'(\theta)\theta}{\eta(\theta)} - (1 - \eta(\theta))\right] \frac{\dot{\theta}}{\theta} \tag{SP1}$$

$$\dot{u} = \lambda(1-u) - \theta q(\theta) u \tag{SP2}$$

In our context, we have that

$$\begin{aligned} \eta(\theta) &= 1 + \frac{q'(\theta)\theta}{q(\theta)} = 1 - \varepsilon \\ \eta'(\theta) &= 0 \end{aligned}$$

Thus, replacing in the planner's conditions, we get

$$\begin{aligned}\dot{\theta} &= \frac{\theta^{1-\varepsilon}}{\varepsilon pc}[(r+\lambda)pc\theta^{\varepsilon} + \varepsilon pc\theta - (1-\varepsilon)(p-z)] \\ \dot{u} &= \lambda - (\lambda + \varepsilon^{1-\varepsilon})u\end{aligned}$$

Let us construct a phase diagram on the (u, θ) space based on the above differential equations and attempt to infer on some properties of the steady state. Clearly, the first equation does not depend on u. Therefore, if we allow for θ to be represented on the y axis, and u on the x axis, we will have a horizontal line representing the locus of points satisfying $\dot{\theta} = 0$. Let θ^* denote the steady state value of this variable. We would like to infer some properties regarding the stability properties of this steady state value. That is, if $\theta > \theta^*$, will it converge to θ^* or not? What if $\theta < \theta^*$? We know that θ^* must be such that

$$0 = \frac{(\theta^*)^{1-\varepsilon}}{\varepsilon pc}[(r+\lambda)pc(\theta^*)^{\varepsilon} + \varepsilon pc\theta^* - (1-\varepsilon)(p-z)]$$

A simple way to study the dynamics of the equilibrium is to log-linearise the differential equation around the steady state value. That is, let $\dot{\theta} = g(\theta)$. Then, a first order expansion yields

$$\dot{\theta} \simeq g(\theta^*) + g'(\theta^*)(\theta - \theta^*)$$

From the fact that $g(\theta^*) = 0$, by definition, we obtain

$$\dot{\theta} \simeq g'(\theta^*)(\theta - \theta^*)$$

Claim 3.4.4. *$g'(\theta^*) > 0$, hence θ^* is unstable.*

Proof. First, it is clear that if $g'(\theta^*) > 0$, then θ^* is unstable. A positive derivative at the steady state means that whenever $\theta > \theta^*$, then $\dot{\theta} > 0$, so that θ increases further and moves further away from its steady state value. The same logic applies to $\theta < \theta^*$, as then $\dot{\theta} < 0$ and we move further away from θ^*. Note that the converse to this claim is also true: if $g'(\theta^*) < 0$, then the steady state is stable.

Now, to show that the derivative is indeed positive, take the derivative at an arbitrary θ

$$g'(\theta) = \frac{r+\lambda}{\varepsilon} + (2-\varepsilon)\theta^{1-\varepsilon} - \frac{(1-\varepsilon)^2(p-z)}{\varepsilon pc}\theta^{-\varepsilon}$$

At the steady state, as we have seen, θ^* satisfies $g(\theta^*) = 0$, or

$$\frac{(1-\varepsilon)(r+\lambda)}{\varepsilon} + (1-\varepsilon)(\theta^*)^{1-\varepsilon} = (\theta^*)^{-\varepsilon}\frac{(1-\varepsilon)^2(p-z)}{\varepsilon pc}$$

Evaluating the derivative at θ^* then leaves us with

$$\begin{aligned}g'(\theta^*) &= \frac{r+\lambda}{\varepsilon} + (2-\varepsilon)(\theta^*)^{1-\varepsilon} - \frac{(1-\varepsilon)(r+\lambda)}{\varepsilon} - (1-\varepsilon)(\theta^*)^{1-\varepsilon} \\ &= r + \lambda + (\theta^*)^{1-\varepsilon} > 0\end{aligned}$$

□

The dynamics for this variable are then summarised in Figure 3.6.

Figure 3.6: Dynamics of θ

What about unemployment, u? Clearly, we have that

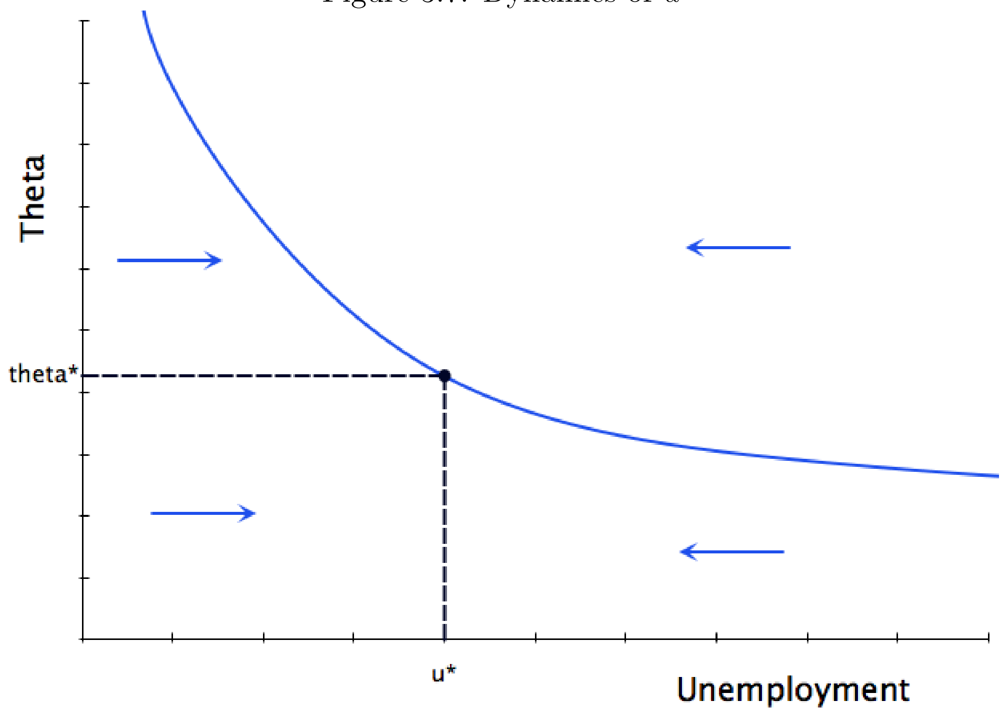
$$\frac{\partial \dot{u}}{\partial u} = -(\lambda + \theta^{1-\varepsilon)} < 0$$

Therefore, unemployment will be stable: whenever u exceeds u^*, it will decrease so as to come back to the steady state, which is given by

$$u^* = \frac{\lambda}{\lambda + \theta^{1-\varepsilon}}$$

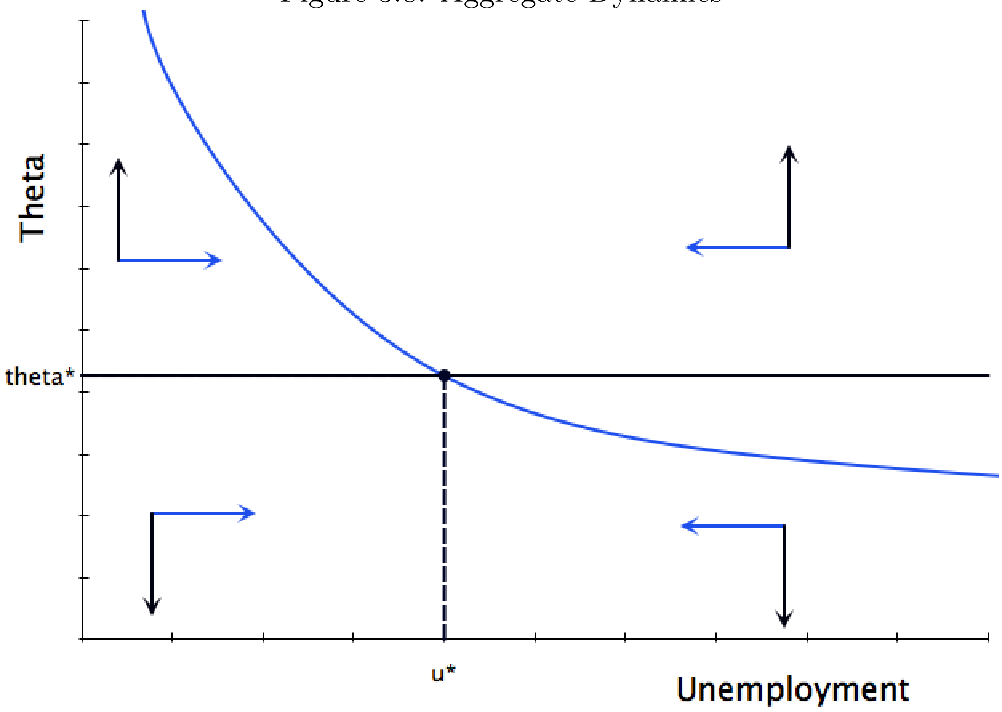
Furthermore, there is a unique value of (steady state) unemployment that is consistent with θ^*. The dynamics of unemployment follow in Figure 3.7.

Figure 3.7: Dynamics of u



What about the aggregate dynamics? As we have seen, due to instability of θ, the steady state (which is unique in this case) appears to be unstable. Note, however, that while u is a stock variable, v is a jump variable. The only predetermined variable in this model is u_0 (the initial level of unemployment). Therefore, in an equilibrium, given u_0, we will observe v_0 immediately adjusting such that $\theta^* = \frac{v_0}{u_0}$. As unemployment moves and adjust towards the steady state, so will vacancies adjust. Therefore, $\theta = \theta^*, \forall t \geq 0$ for an equilibrium to exist. This means that **all adjustment** is made along the $\dot{\theta} = 0$ line, for any initial condition.

Figure 3.8: Aggregate Dynamics



Aggregate Dynamics and the Beveridge Curve

As we have seen, the Beveridge Curve describes a negative relationship between unemployment and vacancies. Imposing steady state in the law of motion of unemployment, and using the assumed functional form for the matching function, we have that

$$u = \frac{\lambda}{\lambda + \left(\frac{v}{u}\right)^{1-\varepsilon}}$$

or

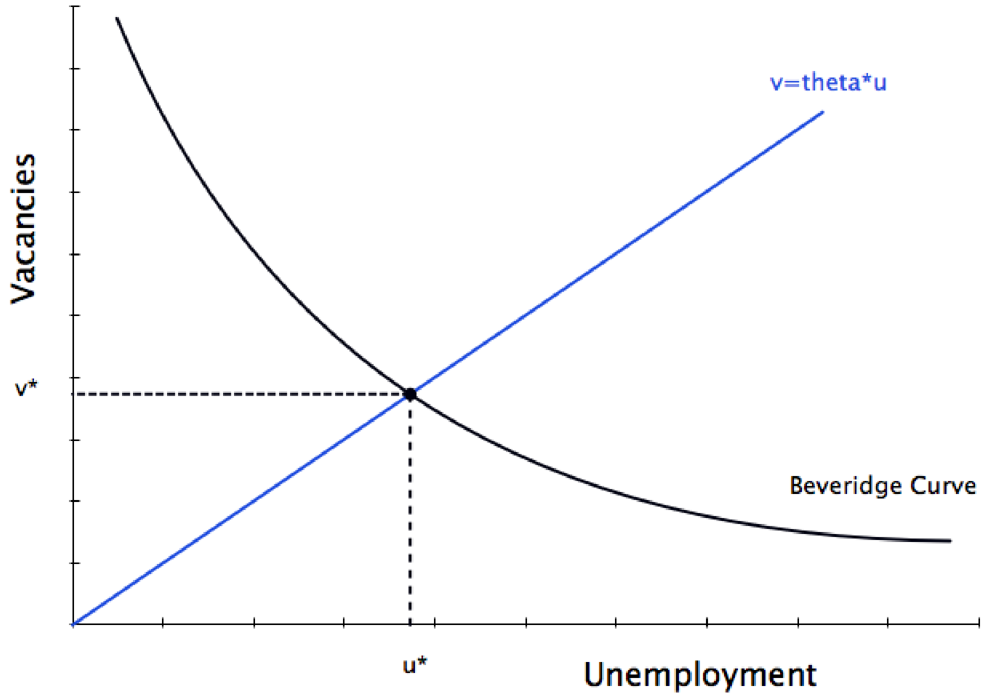
$$v = \left[\frac{\lambda}{u^{\varepsilon}}(1-u)\right]^{\frac{1}{1-\varepsilon}}$$

Similarly, as we have seen, for an equilibrium to be attained in our model, vacancies should immediately react whenever unemployment changes, so that $\theta = \theta^*, \forall t \geq 0$. This induces a linear relationship between the two variables

$$v = \theta^* u$$

where θ^* is some function of the parameters, $\theta^* = f(p, \varepsilon, c, z, r, \lambda)$. The induced relation between the two variables can be graphically represented as in Figure 3.9.

Figure 3.9: Aggregate Dynamics: Beveridge Curve



Thus shocks to the values of parameters can also be analysed using this graphical apparatus. Consider, for example, the case of a productivity shock, $p\uparrow$. As we have seen, in the comparative statics analysis, this induces a rise in θ^*. The following description of the adjustment refers to Figure 3.10 which graphically describes the process. Assume the economy starts at steady state SS0.

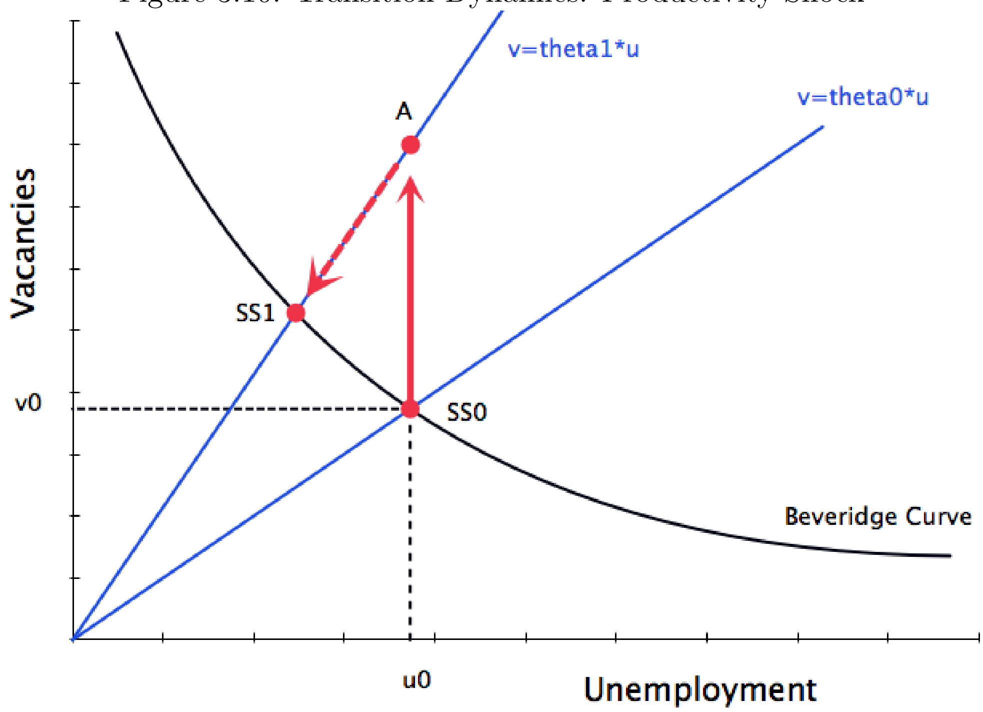
1. $\theta^* \uparrow$ means that the $v = \theta^* u$ locus has just become a steeper line throughout the origin. The economy must immediately jump to the new line, as the equilibrium relation $v = \theta^* u$ must be verified at all times (otherwise, θ diverges and an equilibrium is not possible).

2. What will then adjust so that the economy jumps to the new line? Vacancies? Unemployment? A combination of the two? **Unemployment cannot jump down**: it is a state variable, and must move smoothly (at least downwards). Therefore, vacancies immediately adjust and jump up, so that the economy jumps to point A.

3. Then, it is time for unemployment to start adjusting. Given the rise in θ^*, unemployment must decrease. The adjustment will then take place along the new $v = \theta^* u$ locus as the economy converges to SS1.

4. In the new steady state, unemployment is lower and the number of vacancies is higher, due to the positive productivity shock that the economy experienced. Notice that vacancies overshoot during the adjustment process.

Figure 3.10: Transition Dynamics: Productivity Shock



What if the shock was, instead, negative? Then, the adjustment would be similar, but, this time, vacancies would undershoot: the $v = \theta^* u$ would become less steep and vacancies would jump down. Then, unemployment would slowly increase until the new steady state (with less vacancies and more unemployment) is reached. Notice that the adjustment is always made counter-clockwise in this model (which is consistent with the empirical data on labour market adjustments).

3.4.3 Mortensen and Pissarides with Aggregate Productivity Shocks

The previous discussion motivates a well-known extension of the original M-P article, which features aggregate stochastic productivity shocks. There are two possible states for productivity, $p = \{p_0, p_1\}$, where $p_0 < p_1$. The transition between the two states follows a Poisson process. That is, given some state p_i, the rate of arrival of state p_j is given by δ (thus the conditional distribution is the same for both shocks).

We assume, for simplicity, that $z < p_0 < p_1$. This greatly simplifies the model: as we already know, it will ensure that workers and firms *always* accept matches, regardless of the state. The case in which $p_0 < z$ will be briefly discussed at the end of the section.

Value Functions

The model becomes considerably more complex, as now the value of a filled vacancy, for example, will differ across states. Index the value functions by i to imply that they correspond to the value of such state in aggregate productivity state p_i. Note that, now, we may and will have different values and dynamics for θ and u depending on the productivity state we are in. Therefore, these variables should also be indexed by i.

We explicitly derive the value function for an unfilled vacancy and skip the remaining derivations (as the logic is always the same). Assume, in the discrete formulation, that the aggregate shock is realised between any matches are observed (as usual, this will not matter at all once we move into the realm of continuous time). Thus, with probability $\delta\Delta$, the firm moves to state j and earns continuation value $V_j(t+\Delta)$. With probability $1-\delta\Delta$, it is business as usual on state i. In discrete terms, the value of an unfilled vacancy in state i can be written as

$$V_i(t) = \frac{1}{1+r\Delta}\left\{-p_i c\Delta + \delta\Delta V_j(t+\Delta) + [1-\delta\Delta]\left[q(\theta_i)\Delta J_i(t+\Delta) + (1-q(\theta_i)\Delta)V_i(t+\Delta) + o(\Delta)\right]\right\}$$

Rearranging and dividing by Δ

$$rV_i(t) = -p_i c + \delta[V_j(t+\Delta) - V_i(t+\Delta)] + (1-\delta\Delta)q(\theta_i)[J_i(t+\Delta) - V_i(t+\Delta)] + \frac{V_i(t+\Delta) - V_i(t)}{\Delta} + \frac{o(\Delta)}{\Delta}$$

Let $\Delta \to 0$ and set the time derivative term equal to zero, as we will focus on stationary equilibria, to get

$$rV_i = -p_i c + \delta(V_j - V_i) + q(\theta_i)(J_i - V_i)$$

Similarly, the value of a filled vacancy on state i is

$$rJ_i = p_i - w_i + \delta(J_j - J_i) + \lambda(V_i - J_i)$$

Note that, by indexing w_i to i we are assuming that there is *continuous bargaining in this model*: whenever the state changes, so do the wages. It is not that bargaining occurs when the worker joins the firm, and keeps the same wage forever: such interpretation was compatible with the previous framework, which was fully stationary, but not with the current one.

For the worker, the logic is pretty much the same

$$rU_i = z + \theta_i q(\theta_i)(W_i - U_i) + \delta(U_j - U_i)$$
$$rW_i = w_i + \lambda(U_i - W_i) + \delta(W_j - W_i)$$

Job Creation Condition

Note that, for two states, we are left with a system of **8 Bellman Equations** to be solved for equilibrium.

On a stationary equilibrium, impose free entry at either aggregate state, $V_i = V_j = 0$. From the value function for an unfilled vacancy, this implies that

$$J_i = \frac{p_i c}{q(\theta_i)}, \forall i$$

and from the value function for a filled vacancy, we obtain a relationship between the values in each state

$$J_i = \frac{p_i - w_i + \delta J_j}{r + \delta + \lambda}$$

Substituting J_i, J_j from the expression for the value of an unfilled vacancy, we obtain the new **job creation condition** - in fact, there will be one for each state!

$$\frac{p_i c}{q(\theta_i)} = \frac{p_i - w_i + \delta \frac{p_j c}{q(\theta_j)}}{r + \delta + \lambda}$$

Simplifying:

$$p_i - w_i = (r + \lambda)\frac{p_i c}{q(\theta_i)} + \delta \left(\frac{p_i c}{q(\theta_i)} - \frac{p_j c}{q(\theta_j)} \right)$$

for $i = 1, 2$ and $i \neq j$. Once again, the above condition can be interpreted as a wedge between marginal productivity and magical cost of labour. However, the wedge is now not only affected by the expected hiring cost, but also by the *expected gap in hiring costs* between the two states. Therefore, if the current state has higher average hiring costs, then it will induce a wider gap between productivity and wages.

Wage Determination

The logic is the same, and surrounds solving a Nash Bargaining Problem, now indexed by i, the state. For each state

$$w_i = \arg\max_w (W_i(w) - U)^\beta (J_i(w) - V)^{1-\beta}$$

The FOC, as before, gives us

$$\beta(W_i(w_i) - U_i)^{-1}\frac{\partial W_i}{\partial w_i} + (1-\beta)(J_i(w_i) - V_i)^{-1}\frac{\partial J_i}{\partial w_i} = 0$$

where, from what we have seen

$$\frac{\partial J_i}{\partial w_i} = -\frac{1}{r + \delta + \lambda}$$

and, for the worker, we can write

$$W_i = \frac{w_i + \lambda U_i + \delta W_j}{r + \lambda + \delta} \Rightarrow \frac{\partial W_i}{\partial w_i} = \frac{1}{r + \lambda + \delta}$$

So that, as before, $\frac{\partial J_i}{\partial w_i} = -\frac{\partial W_i}{\partial w_i}$, and we obtain

$$\beta(J_i - V_i) = (1-\beta)(W_i - U_i)$$

Replace $J_i - V_i$ by the expression we found while deriving the job creation condition to get

$$\beta \frac{p_i - w_i + \delta J_j}{r + \delta + \lambda} = (1-\beta)(W_i - U_i)$$

To handle the RHS, recall that

$$W_i = \frac{w_i + \lambda U_i + \delta W_j}{r + \lambda + \delta}$$

So that

$$W_i - U_i = \frac{w_i + \delta W_j - (r+\delta)U_i}{r + \lambda + \delta}$$

Similarly, from the original specification of the value function, we have that

$$(r+\delta)U_i = z + \theta_i q(\theta_i)(W_i - U_i) + \delta U_j$$

The Nash FOC implies that $W_i - U_i = \frac{\beta}{1-\beta}(J_i - V_i)$ (for any state i), so that replacing we obtain

$$(r+\delta)U_i = z + \theta_i q(\theta_i)\frac{\beta}{1-\beta}(J_i - V_i) + \delta U_j$$

Replacing this in the expression for $W_i - U_i$ gives us

$$W_i - U_i = \frac{w_i - z + \delta(W_j - U_j) - \theta_i q(\theta_i)\frac{\beta}{1-\beta}(J_i - V_i)}{r + \lambda + \delta}$$

Replacing $W_j - U_j = \frac{\beta}{1-\beta}(J_j - V_j)$ and $J_i - V_i = \frac{p_i c}{q(\theta_i)}$ (from free-entry) finally leaves us with

$$W_i - U_i = \frac{w_i - z + \delta\frac{\beta}{1-\beta}(J_j - V_j) - \theta_i p_i c\frac{\beta}{1-\beta}}{r + \lambda + \delta}$$

Replacing in the FOC and canceling the denominators

$$\beta(p_i - w_i + \delta J_j) = (1-\beta)\left(w_i - z + \delta\frac{\beta}{1-\beta}(J_j - V_j) - \theta_i p_i c\frac{\beta}{1-\beta}\right)$$

Noting that $V_j = 0$ and simplifying

$$\beta(p_i - w) + \beta\delta J_j = (1-\beta)(w_i - z) + \delta\beta J_j - \theta_i p_i c\beta$$

Rearranging and simplifying

$$w_i = (1-\beta)z + \beta p_i(1 + \theta_i c)$$

Exactly the same expression as before, just that the wage in state i now depends on the productivity and θ at the same state. Note that there is no cross-state dependence on wages, and w_i depends only on variables that refer to the current state (at least directly).

Equilibrium

An equilibrium will now be characterised by

1. The Law of Motion of Unemployment (there is only one)
$$\dot{u} = \lambda(1-u) - \theta_i q(\theta_i)u$$
2. A wage setting equation for each state
$$w_i = (1-\beta)z + \beta p_i(1+\theta_i c), i = 1, 2$$
3. A job creation condition for each state
$$p_i - w_i = (r+\lambda)\frac{p_i c}{q(\theta_i)} + \delta\left(\frac{p_i c}{q(\theta_i)} - \frac{p_j c}{q(\theta_j)}\right), i = 1, 2$$

As before, the wage is fully determined by θ_i. Therefore, we can replace the wage setting equation on the job creation condition to obtain

$$(p_i - z)(1-\beta) - (r + \lambda + \beta\theta_i q(\theta_i))\frac{p_i c}{q(\theta_i)} - \delta\left(\frac{p_i c}{q(\theta_i)} - \frac{p_j c}{q(\theta_j)}\right) = 0$$

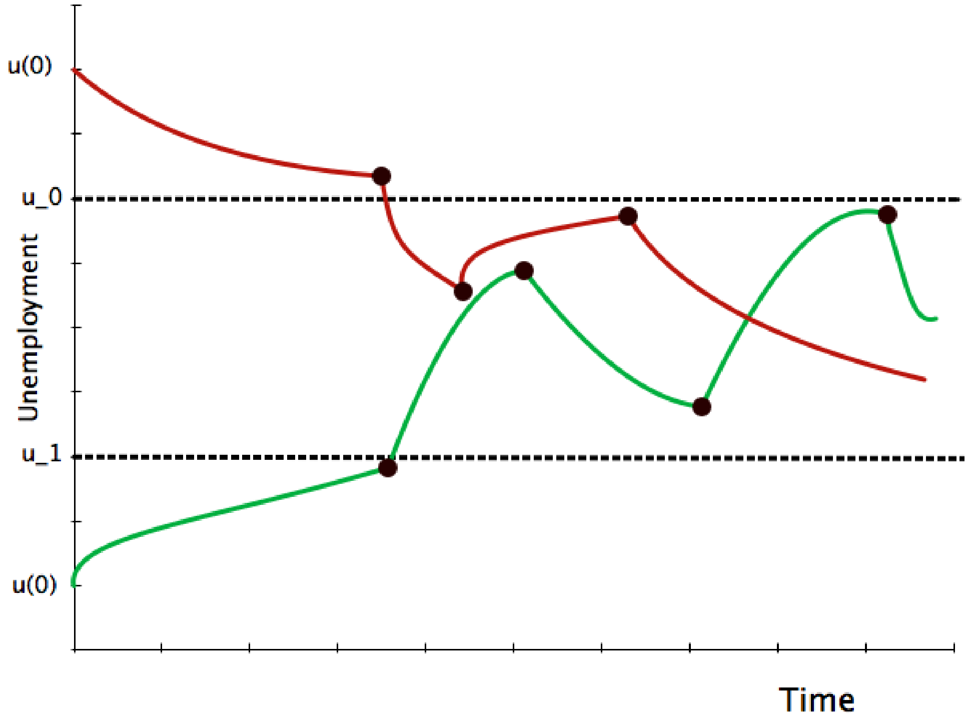
A pair of equations which can be solved for (θ_i, θ_j), the stationary levels of θ **for each aggregate state**.

Equilibrium Dynamics

Even though each productivity state will be associated to a fully stationary steady state (θ_i, u_i), the economy will be moving from one steady state to the other as p_i changes. To see this, let u_0, u_1 denote the steady state levels of unemployment for each productivity level. From prior analysis, we know that p has a negative impact on unemployment. Therefore, $p_0 < p_1 \Rightarrow u_0 > u_1$. Assume we start from some unemployment level $\bar{u} \in [u_1, u_0]$. As we know, unemployment is stable, hence, depending on the productivity state we are in, the economy will approach the corresponding steady state level of unemployment. Say that initial productivity is equal to p_0. Then, $u \to u_0$. As before, θ is always equal to θ_0, and vacancies adjust so that this is the case. When p_0 switches to p_1, the transition will invert its course and unemployment will now converge to u_1. This process will be repeated every time p_i changes. Naturally, it is perfectly possible that the economy will never reach any of the steady state levels of unemployment, and will consistently bump up and down between those two.

Naturally, if $\bar{u} > u_0$ or $\bar{u} < u_1$, unemployment will converge monotonically to the interval $[u_1, u_0]$ regardless of the changes in states, even though, obviously, the current state will affect the *speed* of convergence. To see this, assume that $\bar{u} < u_1$ and the current state is p_0. Then, the economy will approach the stable interval at a greater speed than it would do if the current state were p_1.

Figure 3.11: Equilibrium Dynamics



The model becomes considerably more complicated if we allow for $p_0 < z$. In this case, whenever the economy hits the bad state, workers will prefer to quit their current jobs and unemployment may *jump upwards*.

Chapter 4

Optimal Fiscal Policy with Commitment

4.1 Introduction

In this section, a standard version of the (Walrasian) neoclassical growth model is used to conduct some fiscal policy analysis. In particular, we focus on models of optimal taxation: in this setting, a benevolent fiscal authority needs to finance some exogenous stream of expenditures. It happens, however, that the only available instruments are distortionary. How should the authority set these instruments in an optimal way, so that social welfare is maximised (or the distortions minimised)?

Throughout the analysis, it is assumed that the fiscal authority operates under commitment: that is, when an optimal, state contingent, taxation plan is chosen at $t = 0$, the authority is not allowed to reoptimise after the markets open. This assumption is crucial in the sense that it totally eliminates from the model features such as dynamic inconsistency.

4.2 Environment and Model Set-up

As mentioned, the underlying model is a neoclassical stochastic growth model with endogenous labour supply.

1. Time is discrete, agents are homogeneous and infinitely-lived. This allows us to focus on a representative agent.

2. There is uncertainty, modeled each period through dependence on an underlying state or set of states that belong to a finite set, $s_t \in S = \{s_1, \ldots, s_N\}$. A sequence of states, or *history* is denoted as $s^t = (s_0, \ldots, s_t)$. The probability distribution

associated with each history may be time-varying and is denoted by $pi_t(s^t)$. The initial state s_0 is known to the fiscal authority when choosing the optimal taxation plans, before markets open.

3. There are two goods: labour and a consumption good that may also be used for investment.

4. Government expenditures are exogenous and expressed in terms of the consumption good, $\{g_t(s^t)\}_{t=0}^{\infty}$.

5. The economy's feasibility constraint is standard, and notation is set such that capital chosen as part of current investment is denoted by $k_t(s^t)$. This means that capital used in production at t is $k_{t-1}(s^{t-1})$

$$c_t(s^t) + g_t(s^t) + k_t(s^t) = F[k_{t-1}(s^{t-1}), l_t(s^t), s_t] + (1-\delta)k_{t-1}(s^{t-1})$$

   Note that the production function is allowed to depend directly on s_t, the current state (through a technology shock, for example).

6. Preferences are given by

$$U = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u[c_t(s^t), l_t(s^t)]$$

   where $\beta \in (0,1)$, S^t is the set of all possible histories up to time t, u is strictly increasing, strictly concave and three times continuously differentiable. Inada conditions ensure that $c_t(s^t) > 0$ and $l_t(s^t) \in (0,1)$, for simplicity, so that we do not have to worry about interiority of solutions.

7. The government can use three different instruments to finance the exogenous stream of expenditures:

   (a) Proportional labour income taxes, $\tau_t(s^t)$

   (b) Proportional taxes on capital income $\theta_t(s^t)$

   (c) Public debt $b_t(s^t)$

   The government budget constraint, which we will shortly present, must be satisfied for all t. This means that the government will only be able to freely set two of the three instruments at any given point in time, as the third will be the residual and determined by the other two plus expenditures.

Given the description of the environment, and letting $w_t(s^t), r_t(s^t), R_t^b(s^t)$ denote wages, the net interest rate on capital and the gross interest rate on public debt, respectively, the sequential budget constraint for the representative agent can be written as

$$c_t(s^t)+k_t(s^t)+b_t(s^t) = [1-\tau_t(s^t)]w_t(s^t)l_t(s^t)+[1+(1-\theta_t(s^t))(r_t(s^t)-\delta)]k_{t-1}(s^{t-1})+R_t^b(s^t)b_{t-1}(s^{t-1})$$

Notice that $\theta_t(s^t)$ taxes capital income *net of depreciation.* For notational simplicity, let us define a *gross effective interest rate on capital* as

$$R^k_t(s^t) := [1 + (1 - \theta_t(s^t))(r_t(s^t) - \delta)]$$

The additional constraints imposed on the representative agent are:

1. $c_t(s^t) \geq 0, l_t(s^t) \in [0, 1]$, which will be trivially satisfied thanks to the Inada conditions.
2. $k_t(s^t) \geq 0$, there is no capital borrowing in the model.
3. $b_t(s^t) \in [-B, B^G]$, there is an exogenous borrowing limit on debt imposed on the agent, B. Furthermore, the government faces a borrowing constraint equal to B^G, so that this is the total maximum amount of public debt that can be outstanding in the economy.

4.3 Describing the Equilibrium

4.3.1 Competitive Equilibrium

Some familiar definitions are adapted to this particular setting. We consider the traditional definition of competitive equilibrium that is silent on how fiscal policy is pursued, and just presupposes that a public budget constraint is satisfied.

Definition 4.3.1. *An* ***allocation*** $X = \{x_t(s^t)\}^{\infty}_{t=0,s^t \in S^t}$ *is a sequence of collections of quantities, where*

$$x_t(s^t) = [c_t(s^t), l_t(s^t), k_t(s^t), b_t(s^t)]$$

Definition 4.3.2. *A* ***price system*** $P = \{p_t(s^t)\}^{\infty}_{t=0,s^t \in S^t}$ *is a sequence of prices*

$$p_t(s^t) = [w_t(s^t), r_t(s^t), R^b_t(s^t)]$$

The government must satisfy a budget constraint, taking the stream of expenditures as given. This constraint must be satisfied at any period and node s_t

$$g_t(s^t) + R^b_t(s^t)b_{t-1}(s^{t-1}) = b_t(s^t) + \tau_t(s^t)w_t(s^t)l_t(s^t) + \theta_t(s^t)[r_t(s^t) - \delta]k_{t-1}(s^{t-1})$$

where the LHS presents total expenditures (exogenous expenditure plus principal and interest payments on outstanding public debt), and the RHS corresponds to public revenues: bond issuance, labour income taxes and capital income taxes.

Definition 4.3.3. *A* ***government tax policy*** *is a sequence* $\phi = \{\tau_t(s^t), \theta_t(s^t)\}^{\infty}_{t=0,s^t \in S^t}$.

Note that public debt is implicitly considered as the residual fiscal instruments, whose adjustment ensures that the public budget constraint is satisfied at all times.

We proceed by presenting the usual definition of equilibrium:

Definition 4.3.4. *A **Competitive Equilibrium** is a government tax policy ϕ, an allocation X and a price system P such that*

1. *Given ϕ, P, the allocation X solves the household problem.*
2. *Given ϕ, P, firms optimise by solving a static problem*

$$r_t(s^t) = F_k[k_{t-1}(s^{t-1}), l_t(s^t), s_t]$$
$$w_t(s^t) = F_l[k_{t-1}(s^{t-1}), l_t(s^t), s_t]$$

3. *The goods market, labour market and bonds market clear.*
4. *The government budget constraint is satisfied.*

4.3.2 Ramsey Equilibrium

We now proceed by slightly tweaking the traditional definition so as to bring it to the realm of optimal taxation policy. The whole point of the Ramsey Problem (=optimal taxation problem) is, as explained, for the government to finance its expenditures by causing as much little distortion as possible over private decisions. This suggests that the fiscal authority will solve a problem not unlike that of a social planner who seeks to find a constrained optimal allocation: a socially efficient allocation that can be supported as an equilibrium given the physical/technological constraints faced by the economy.

Therefore, the government will want to choose a tax policy ϕ, but accounting for the fact that people's decisions will change upon different choices of ϕ. Therefore, different equilibria may emerge. The following definitions formalise this notion of dependence.

Definition 4.3.5. *An **allocation rule** is a sequence of functions*

$$X(\phi) = \{x_t(s^t|\phi)\}_{t=0,s^t \in S^t}^{\infty}$$

Definition 4.3.6. *A **price rule** is a sequence of functions*

$$P(\phi) = \{w_t(s^t|\phi), r_t(s^t|\phi), R_t^b(s^t|\phi)\}_{t=0,s^t \in S^t}^{\infty}$$

Leading us to the main object of interest in this section:

Definition 4.3.7. *A **Ramsey Equilibrium** is a policy ϕ, an allocation rule $X(\phi)$ and a price rule $P(\phi)$ such that:*

1. *The policy ϕ solves the Ramsey Problem*

$$\max_{\phi} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) u[c_t(s^t|\phi), l_t(s^t|\phi)]$$

   subject to

$$g_t(s^t)+R_t^b(s^t|\phi)b_{t-1}(s^{t-1}|\phi) = b_t(s^t|\phi)+\tau_t(s^t|\phi)w_t(s^t|\phi)l_t(s^t|\phi)+\theta_t(s^t|\phi)[r_t(s^t|\phi)-\delta]k_{t-1}(s^{t-1}|\phi)$$

   and $k_{-1}, b_{-1}, \theta_0(s_0)$ given.

2. *For every ϕ, the allocation and price rules $X(\phi), P(\phi)$ constitute a Competitive Equilbrium.*

This formalises the notion that the government chooses, at $t = 0$, before markets open, an optimal sequence of fiscal instruments that maximises the present discounted value of utility for the representative consumer. Moreover, in the constrained optimality spirit of the whole problem, the government must choose sequences that induce competitive equilibria, thus taking the distortionary equilibrium effect of its own tax choices into account when setting the instruments.

Note that the whole point of the problem is that the government *has more than one possible tax instrument* available for financing expenditures. If only one instrument were available, policy could be discretionary and still constrained Pareto optimal. Thus the whole problem arises from a nontrivial choice between tax instruments.

By imposing commitment, we are ruling out issues such as dynamic inconsistency: the government chooses a sequence of taxes at $t = 0$, and cannot do anything else once markets open. Note that ϕ is a complete state-contingent tax plan. Commitment is important because it prevents the government from undertaking actions that, while seemingly optimal in a static context, are not so in a dynamic setting. Once markets open, agents take decision and prices are formed, the government could be tempted to change its own course of action. For example, lowering labour taxes and raising capital taxes if too much savings is perceived in the economy. This is the point of the second part of the definition: by forcing the government to acknowledge that $X(\phi), P(\phi)$ must constitute equilibria along with ϕ. Thus the last condition effectively imposes *subgame perfection Chari and Kehoe (1998) discuss this in detail: if the last condition is not imposed, then we are effectively not imposing optimality on off-equilibrium paths and the set of equilibria becomes much larger, including arbitrary equilibria that make no sense at all.*

It is also worth discussing the not innocuous assumption that $\theta_0(s_0)$, the initial tax on capital income, is given. This is done to ensure that the economy can actually *start off.* If the government had control of θ_0, and k_{-1} is given, it could be tempted to fully tax capital in period 0. Why? Since initial capital is given, by raising the capital income tax the government would, in effect, be levying a *lump-sum* tax (as this would no distort the agents' decisions). Therefore, the government could raise a non-distortionary tax today, save its proceedings and reduce usage of distortionary taxation in future periods.

However, the government may be tempted to set this tax to such a level that capital is fully taxed. This would imply no investment and, therefore, no production in subsequent periods - the economy would be trapped in a zero production equilibrium. Further note that by choosing θ and τ, we are giving the government control of *effective* factor prices and, through that, control over demand and supply of each factor of production.

Equilibrium Conditions

Let p_t denote the shadow price of consumption in each period, the Lagrange Multiplier associated with the household's budget constraint at each period. It can be easily checked that the following equations fully describe a competitive equilibrium

$$c_t(s^t)+k_t(s^t)+b_t(s^t) = [1-\tau_t(s^t)]w_t(s^t)l_t(s^t)+[1+(1-\theta_t(s^t))(r_t(s^t)-\delta)]k_{t-1}(s^{t-1})+R_t^b(s^t)b_{t-1}(s^{t-1}) \tag{HBC}$$

$$\beta^t u_c(s^t)\pi_t(s^t) - p_t(s^t) \leq 0 \tag{CFOC}$$

$$\beta^t u_l(s^t)\pi_t(s^t) + p_t(s^t)[1-\tau_t(s^t)]w_t(s^t) \leq 0 \tag{LFOC}$$

$$\left[p_t(s^t) - \sum_{s_{t+1}|s^t} p_{t+1}(s^{t+1})R_{t+1}^b(s^{t+1})\right] b_t(s^t) = 0 \tag{BCS}$$

$$\left[p_t(s^t) - \sum_{s_{t+1}|s^t} p_{t+1}(s^{t+1})R_{t+1}^k(s^{t+1})\right] k_t(s^t) = 0 \tag{KCS}$$

$$\lim_{t\to\infty} \sum_{s^t} p_t(s^t)b_t(s^t) = 0 \tag{BTC}$$

$$\lim_{t\to\infty} \sum_{s^t} p_t(s^t)k_t(s^t) = 0 \tag{KTC}$$

$$r_t(s^t) = F_k[k_{t-1}(s^{t-1}), l_t(s^t), s_t] \tag{KF}$$

$$w_t(s^t) = F_l[k_{t-1}(s^{t-1}), l_t(s^t), s_t] \tag{LF}$$

$$g_t(s^t) + R_t^b(s^t)b_{t-1}(s^{t-1}) = b_t(s^t) + \tau_t(s^t)w_t(s^t)l_t(s^t) + \theta_t(s^t)[r_t(s^t)-\delta]k_{t-1}(s^{t-1}) \tag{GBC}$$

where

1. (HBC) is the household's budget constraint
2. (CFOC) and (LFOC) are the household's FOC with respect to consumption and labour
3. (BCS) and (KCS) are the complementary slackness/non-arbitrage conditions with respect to bonds and capital emanating from the household's problem. They can also be seen as Euler Equations
4. (BTC) and (KTC) are the necessary and sufficient transversality conditions with respect to bonds and capital

5. (KF) and (LF) are the firm's FOC
6. (GBC) is the government's budget constraint

and u_c, u_l denote the marginal utility of consumption and disutility of working, respectively.

4.4 The Primal Approach

The Ramsey Problem is typically difficult to solve in its pure form, as presented above. The government would have to maximise the present discounted value of utility subject to the full set of equilibrium conditions that were outlined in the previous section.

An alternative approach to the Ramsey Problem involves consideration that, by choosing taxes, as already remarked, the government can effectively control effective factor prices and, therefore, demand and supply of all goods that are traded in this economy. This gives rise to the *Primal Approach*, a setting where prices and taxes are eliminated from the problem, and the government chooses allocations directly. Thus the government effectively solves the Ramsey Problem, but instead of being constrained to a full set of equilibrium conditions, it needs only to consider feasibility and an additional condition that, as we will show, ensures that the chosen allocations are compatible with a competitive equilibrium. These two constraints will be the typical **feasibility constraint** and, the second, that ensures that the allocations are compatible with an equilibrium, will be called **implementability constraint**.

The name of this method is chosen to contrast with the original *dual approach*, in which the government chooses taxes directly and must account for their impact on allocations and prices.

Equivalence between the two approaches is established by the following result:

Proposition 4.4.1. *The allocations in a competitive equilibrium satisfy the feasibility constraint*

$$c_t(s^t) + g_t(s^t) + k_t(s^t) = F[k_{t-1}(s^{t-1}), l_t(s^t), s_t] + (1-\delta)k_{t-1}(s^{t-1})$$

and the implementability constraint

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)\{u_c(s^t)c_t(s^t) + u_l(s^t)l_t(s^t)\} = u_c(s_0)[R_0^k(s_0)k_{-1} + T_0^b(s_0)b_{-1}]$$

where

$$R_0^k(s_0) = 1 + [1-\theta_0(s_0)][F_k(k_{-1}, l_0(s_0), s_0) - \delta]$$

Furthermore, given allocations and time 0 *policies that satisfy feasibility and implementability, we can construct policies, prices and debt holdings which, together with the given allocations and time* 0 *policies, constitute a competitive equilibrium.*

The spirit of the proposition is the following: for any allocation that satisfies feasibility and implementability, there is a competitive equilibrium from where prices and fiscal polices can be 'backed out'. Note that both constraints can, in some sense, be seen as budget constraints: while feasibility ensures that the allocation is *physically feasible*, in the technological sense (as in all resources that are consumed or employed exist/are produced), implementability ensures that whatever equilibrium is generated, the representative agent's budget constraint is satisfied. After all, and as we shall see in the proof, implementability is almost a time zero budget constraint, where prices have been substituted using first order conditions, both on the firm and the agent's side.

Proof. We first show that the allocations in a competitive equilibrium satisfy feasibility and implementability. Feasibility is almost immediate. Take (HBC) and (GBC). Add both sides in order to get (I will omit the state argument momentarily for notational ease)

$$c_t + k_t + b_t + g_t + R_t^b b_{t-1} = (1-\tau_t)w_t l_t + R_t^k k_{t-1} + R_t^b b_{t-1} + \tau_t w_t l_t + b_t + \theta_t(r_t - \delta)k_{t-1}$$

Cancel all common terms and replace for R_t^k to obtain

$$c_t + k_t + g_t = w_t l_t + (1 + r_t - \delta)k_{t-1}$$

From (KF) and (LF), and the fact that the neoclassical production function is homogeneous of degree one, we obtain

$$c_t + k_t + g_t = F(k_{t-1}, l_t, s_t) + (1-\delta)k_{t-1}$$

as we wanted.

To show that a competitive equilibrium allocation satisfies implementability, first multiply (HBC) by the shadow price $p_t(s^t)$ and rearrange to obtain

$$0 = p_t(s^t)[c_t(s^t) - [1-\tau_t(s^t)]w_t(s^t)l_t(s^t) + k_t(s^t) + b_t(s^t) - R_t^k(s^t)k_{t-1}(s^{t-1}) - R_t^b(s^t)b_{t-1}(s^{t-1})]$$

Now, sum the above expression over all periods and possible histories. Since the above budget constraint must equal to zero at each node s^t, so must the sum equal zero

$$0 = \sum_{t=0}^{\infty}\sum_{s^t} p_t(s^t)[c_t(s^t) - [1-\tau_t(s^t)]w_t(s^t)l_t(s^t) + k_t(s^t) + b_t(s^t) - R_t^k(s^t)k_{t-1}(s^{t-1}) - R_t^b(s^t)b_{t-1}(s^{t-1})]$$

In particular, break down the above sum in its *consumption* and *investment* parts as

$$\begin{aligned} 0 = &\sum_{t=0}^{\infty}\sum_{s^t} p_t(s^t)[c_t(s^t) - [1-\tau_t(s^t)]w_t(s^t)l_t(s^t)] \\ &+ \lim_{T\to\infty}\sum_{t=0}^{T}\sum_{s^t} p_t(s^t)[k_t(s^t) + b_t(s^t) - R_t^k(s^t)k_{t-1}(s^{t-1}) - R_t^b(s^t)b_{t-1}(s^{t-1})] \end{aligned}$$

From the investment part, separate the terms concerning period $t = 0$ from the rest

$$
\begin{aligned}
0 = & \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t)[c_t(s^t) - [1 - \tau_t(s^t)]w_t(s^t)l_t(s^t)] \\
& + p_0(s_0)[k_0(s_0) + b_0(s_0) - R_0^k(s_0)k_{-1} - R_0^b(s_0)b_{-1}] \\
& + \lim_{T\to\infty} \sum_{t=1}^{T} \sum_{s^t} p_t(s^t)[k_t(s^t) + b_t(s^t) - R_t^k(s^t)k_{t-1}(s^{t-1}) - R_t^b(s^t)b_{t-1}(s^{t-1})]
\end{aligned}
$$

We now want to group all capital and bond terms in such a way that they have the same timing. Ignore the terms timed at $t = -1$, as the agent has no control over them. Take, for example, all terms relating to physical capital. We can rewrite the sum over time and states as (ignoring the limit term for notational simplicity)

$$
\begin{aligned}
p_0(s_0)k_0(s^0) + \sum_{t=1}^{T} \sum_{s^t} p_t(s^t)k_t(s^t) - \sum_{t=1}^{T} \sum_{s^t} p_t(s^t)R_t^k(s^t)k_{t-1}(s^{t-1}) \\
= \sum_{t=0}^{T} \sum_{s^t} p_t(s^t)k_t(s^t) - \sum_{t=1}^{T} \sum_{s^t} p_t(s^t)R_t^k(s^t)k_{t-1}(s^{t-1}) \\
= \sum_{t=0}^{T} \sum_{s^t} p_t(s^t)k_t(s^t) - \sum_{t=0}^{T-1} \sum_{s^{t+1}} p_{t+1}(s^{t+1})R_{t+1}^k(s^{t+1})k_t(s^t)
\end{aligned}
$$

while the first step simply grouped the $t = 0$ term with the first summation, the second step cunningly changed the indices of the summation: instead of summing current prices times past capital, we have switched notation such that we now sum future prices and current capital. This allows us to use capital with the same timing everywhere in the expression. The tricks have not ended: note that summing over all histories at $t + 1$ is exactly the same as summing over all histories at t and then summing over all possible nodes that may follow. That is

$$
\sum_{s^{t+1}} = \sum_{s^t} \sum_{s_{t+1}|s^t}
$$

akin to conditioning. Apply this transformation to write the previous term as

$$
\begin{aligned}
& \sum_{t=0}^{T} \sum_{s^t} p_t(s^t)k_t(s^t) - \sum_{t=0}^{T-1} \sum_{s^t} \sum_{s_{t+1}|s^t} p_{t+1}(s^{t+1})R_{t+1}^k(s^{t+1})k_t(s^t) \\
= & \sum_{t=0}^{T} \sum_{s^t} \left[p_t(s^t) - \sum_{s_{t+1}|s^t} p_{t+1}(s^{t+1})R_{t+1}^k(s^{t+1}) \right] k_t(s^t) + \sum_{s^T} p_T(s^T)R_T^k(s^T)k_T(s^T)
\end{aligned}
$$

Do this not only for capital, but also for bonds, in our time zero budget constraint to obtain the following expression

$$
\begin{aligned}
0 = & \sum_{t=0}^{\infty}\sum_{s^t} p_t(s^t)[c_t(s^t) - [1-\tau_t(s^t)]w_t(s^t)l_t(s^t)] \\
& - p_0(s_0)[R_0^k(s_0)k_{-1} + R_0^b(s_0)b_{-1}] \\
& + \lim_{T\to\infty}\sum_{t=0}^{T}\sum_{s^t}\left[p_t(s^t) - \sum_{s_{t+1}|s^t} p_{t+1}(s^{t+1})R_{t+1}^k(s^{t+1})\right]k_t(s^t) \\
& + \lim_{T\to\infty}\sum_{s^T} p_T(s^T)R_T^k(s^T)k_T(s^T) \\
& + \lim_{T\to\infty}\sum_{t=0}^{T}\sum_{s^t}\left[p_t(s^t) - \sum_{s_{t+1}|s^t} p_{t+1}(s^{t+1})R_{t+1}^b(s^{t+1})\right]b_t(s^t) \\
& + \lim_{T\to\infty}\sum_{s^T} p_T(s^T)R_T^b(s^T)b_T(s^T)
\end{aligned}
$$

From (BCS), (KCS) (the Euler Equations), the third and fifth lines are equal to zero. Also, from (BTC), (KTC), the fourth and sixth lines are zero as well [1]. Thus we are left with

$$\sum_{t=0}^{\infty}\sum_{s^t} p_t(s^t)[c_t(s^t) - [1-\tau_t(s^t)]w_t(s^t)l_t(s^t)] = p_0(s_0)[R_0^k(s_0)k_{-1} + R_0^b(s_0)b_{-1}]$$

From (CFOC) and (LFOC) we know that

$$
\begin{aligned}
p_t(s^t) &= \beta^t\pi_t(s^t)u_c(s^t) \\
p_0(s^0) &= u_c(s_0) \\
-(1-\tau_t(s^t))w_t(s^t)p_t(s^t) &= \beta^t\pi_t(s^t)u_l(s^t)
\end{aligned}
$$

so that replacing, we obtain

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t(s^t)[u_c(s^t)c_t(s^t) + u_l(s^t)l_t(s^t)] = u_c(s_0)[R_0^k(s_0)k_{-1} + R_0^b(s_0)b_{-1}]$$

which is exactly the implementability constraint, as we wanted to show.

It now remains to show that from any allocations satisfying feasibility and implementability, we can back out prices, tax policies and debt holdings that are consistent with a competitive equilibrium.

To see this, take some real allocation $\{c_t(s^t), l_t(s^t), k_t(s^t)\}_{t=0}^{\infty}$ satisfying both feasibility and implementability, given initial conditions $(k_{-1}, b_{-1}, R_0^k, R_0^b)$ and an exogenous sequence of expenditures $\{g_t(s^t)\}_{t=0}^{\infty}$. We seek to construct a competitive equilibrium by

[1] Clearly, none of these expressions can be negative in equilibrium. Given that effective interest rates are bounded, it can then be shown that the transversality conditions imply that those two lines will be zero.

'constructing' the supporting sequences $\{b_t, w_t, r_t, R_t^b, \tau_t, \theta_t\}$ that compose a well-defined equilibrium.

We start by constructing bond holdings, b_t. To do this, fix some arbitrary node s^r. Take the agent's budget constraint at s^r, multiply both sides by the shadow price $p_t(s^t)$ and sum over forward nodes, starting with the immediate successor node of s^r

$$\sum_{t=r+1}^{\infty}\sum_{s^t|s^r} p_t(s^t)[c_t(s^t)+k_t(s^t)+b_t(s^t)] = \sum_{t=r+1}^{\infty}\sum_{s^t|s^r} p_t(s^t)[(1-\tau_t(s^t))w_t(s^t)l_t(s^t) + R_t^b(s^t)b_{t-1}(s^{t-1}) + R_t^k(s^t)k_{t-1}(s^{t-1})]$$

Isolate the financial sector from the 'consumption' sector

$$\begin{aligned}
&\sum_{t=r+1}^{\infty}\sum_{s^t|s^r} p_t(s^t)[c_t(s^t) - (1-\tau_t(s^t))w_t(s^t)l_t(s^t)] = \\
&= -\lim_{T\to\infty}\sum_{t=r+1}^{T}\sum_{s^t|s^r} p_t(s^t)[k_t(s^t)+b_t(s^t)] \\
&+ \lim_{T\to\infty}\sum_{t=r+1}^{T}\sum_{s^t|s^r} p_t(s^t)[R_t^b(s^t)b_{t-1}(s^{t-1}) + R_t^k(s^t)k_{t-1}(s^{t-1})]
\end{aligned}$$

Take the second term on the RHS of the budget constraint and decompose it in the two terms: the immediate successor node of s^r and the remaining nodes

$$\begin{aligned}
&\lim_{T\to\infty}\sum_{t=r+1}^{T}\sum_{s^t|s^r} p_t(s^t)[R_t^b(s^t)b_{t-1}(s^{t-1}) + R_t^k(s^t)k_{t-1}(s^{t-1})] = \\
&= \sum_{s_{r+1}|s^r} p_{t+1}(s^{t+1})[R_{r+1}^b(s^{r+1})b_r(s^r) + R_{r+1}^k(s^{r+1})k_r(s^r)] \\
&+ \lim_{T\to\infty}\sum_{t=r+2}^{T}\sum_{s^t|s^r} p_t(s^t)[R_t^b(s^t)b_{t-1}(s^{t-1}) + R_t^k(s^t)k_{t-1}(s^{t-1})]
\end{aligned}$$

Also, the first term on the RHS of the budget constraint can be decomposed as

$$\lim_{T\to\infty}\sum_{t=r+1}^{T}\sum_{s^t|s^r} p_t(s^t)[k_t(s^t)+b_t(s^t)] = \lim_{T\to\infty}\sum_{t=r+1}^{T-1}\sum_{s^t|s^r} p_t(s^t)[k_t(s^t)+b_t(s^t)] + \lim_{T\to\infty}\sum_{s^T|s^r} p_T(s^T)[k_T(s^T)+b_T(s^T)]$$

Capital is always non-negative, due to the borrowing constraint, and we would like to construct an equilibrium in which the government does not keep resources for itself (which is clearly suboptimal from a welfare point of view). Thus we seek to have $b_T \geq 0$, which implies that

$$\lim_{T\to\infty} \sum_{s^T|s^r} p_T(s^T)[k_T(s^T) + b_T(s^T)] \geq 0$$

In order not to miss any history, sum over s^r to obtain

$$\lim_{T\to\infty} \sum_{s^T} p_T(s^T)[k_T(s^T) + b_T(s^T)] \geq 0$$

For the Transversality Conditions to hold, the above term must be exactly equal to zero. Having established that $k_T, b_T \geq 0$, and wishing to impose the Transversality Conditions, we have that

$$\lim_{T\to\infty} \sum_{s^T} p_T(s^T) b_T(s^T) = 0$$
$$\lim_{T\to\infty} \sum_{s^T} p_T(s^T) k_T(s^T) = 0$$

hence

$$\lim_{T\to\infty} \sum_{s^T} p_T(s^T)[k_T(s^T) + b_T(s^T)] = 0$$

Now, impose the above condition and take all the decompositions back to the original budget constraint, writing it as

$$\begin{aligned}
&\sum_{t=r+1}^{\infty} \sum_{s^t|s^r} p_t(s^t)[c_t(s^t) - (1-\tau_t(s^t))w_t(s^t)l_t(s^t)] = \\
&= -\lim_{T\to\infty} \sum_{t=r+1}^{T-1} \sum_{s^t|s^r} p_t(s^t)[k_t(s^t) + b_t(s^t)] \\
&+ \sum_{s_{r+1}|s^r} p_{t+1}(s^{t+1})[R^b_{r+1}(s^{r+1})b_r(s^r) + R^k_{r+1}(s^{r+1})k_r(s^r)] \\
&+ \lim_{T\to\infty} \sum_{t=r+2}^{T} \sum_{s^t|s^r} p_t(s^t)[R^b_t(s^t)b_{t-1}(s^{t-1}) + R^k_t(s^t)k_{t-1}(s^{t-1})]
\end{aligned}$$

Use the 'conditioning trick' to combine the first and third terms on the RHS

$$
\begin{aligned}
&\sum_{t=r+1}^{\infty}\sum_{s^t|s^r} p_t(s^t)[c_t(s^t) - (1-\tau_t(s^t))w_t(s^t)l_t(s^t)] = \\
&= -\lim_{T\to\infty}\sum_{t=r+1}^{T-1}\sum_{s^t|s^r}\left[p_t(s^t) - \sum_{s^{t+1}|s^t} p_{t+1}(s^{t+1})R^k_{t+1}(s^{t+1})\right]k_t(s^t) \\
&- \lim_{T\to\infty}\sum_{t=r+1}^{T-1}\sum_{s^t|s^r}\left[p_t(s^t) - \sum_{s^{t+1}|s^t} p_{t+1}(s^{t+1})R^b_{t+1}(s^{t+1})\right]b_t(s^t) \\
&+ \sum_{s_{r+1}|s^r} p_{r+1}(s^{r+1})[R^b_{r+1}(s^{r+1})b_r(s^r) + R^k_{r+1}(s^{r+1})k_r(s^r)]
\end{aligned}
$$

In order to be consistent with a competitive equilibrium, we need the Euler Equations to hold. Hence the second and third lines should all be equal to zero. This leaves us with

$$
\begin{aligned}
&\sum_{t=r+1}^{\infty}\sum_{s^t|s^r} p_t(s^t)[c_t(s^t) - (1-\tau_t(s^t))w_t(s^t)l_t(s^t)] \\
&= \sum_{s_{r+1}|s^r} p_{r+1}(s^{r+1})[R^b_{r+1}(s^{r+1})b_r(s^r) + R^k_{r+1}(s^{r+1})k_r(s^r)] =
\end{aligned}
$$

But, from the Euler Equations, we can rewrite the second line as

$$
\begin{aligned}
&\sum_{s_{r+1}|s^r} p_{r+1}(s^{r+1})[R^b_{r+1}(s^{r+1})b_r(s^r) + R^k_{r+1}(s^{r+1})k_r(s^r)] = \\
&= b_r(s^r)\sum_{s_{r+1}|s^r} p_{r+1}(s^{r+1})R^b_{r+1}(s^{r+1}) + k_r(s^r)\sum_{s_{r+1}|s^r} p_{r+1}(s^{r+1})R^k_{r+1}(s^{r+1}) = \\
&= k_r(s^r)p_r(s^r) + b_r(s^r)p_r(s^r)
\end{aligned}
$$

Finally leaving us with a much more simplified expression for the budget constraint

$$
\sum_{t=r+1}^{\infty}\sum_{s^t|s^r} p_t(s^t)[c_t(s^t) - (1-\tau_t(s^t))w_t(s^t)l_t(s^t)] = k_r(s^r)p_r(s^r) + b_r(s^r)p_r(s^r)
$$

Furthermore, from the FOC for the consumer

$$
\begin{aligned}
-(1-\tau_t(s^t))w_t(s^t)l_t(s^t) &= \frac{\beta^t u_l(s^t)\pi_t(s^t)}{p_t(s^t)} \\
p_t(s^t) &= \beta^t \pi_t(s^t) u_c(s^t)
\end{aligned}
$$

Replacing in our condition then leaves us with

$$b_r(s^r) = -k_r(s^r) + \frac{1}{\beta^r u_c(s^r)\pi_r(s^r)} \lim_{T\to\infty} \sum_{t=r+1}^{T} \sum_{s^t|s^r} \beta^t \pi_t(s^t)[u_c(s^t)c_t(s^t) + u_l(s^t)l_t(s^t)]$$

Thus we have managed to construct, for any arbitrary node, a sequence of bond holdings as a function, only, of real allocations (which we have by assumptions). That is, $\{b_t(s^t)\}$ depends only on the model's primitives and our initial raw materials, $\{c_t, l_t, k_t\}$!

Now note that $\{r_t(s^t), w_t(s^t)\}_{t=0}^{\infty}$ can be easily constructed from the real allocations using the firm's FOC, as a function of $\{k_t, l_t\}$. It remains then to determine $(R_t^b, \theta_t, \tau_t)$. Labour income taxes can be extracted from the FOC for labour, from the household's problem. After replacing for $p_t(s^t)$, we get

$$\tau_t(s^t) = \frac{u_l(s^t) + w_t(s^t)u_c(s^t)}{w_t(s^t)u_c(s^t)}$$

where all objects are already known (including wages).

Thus the last step is to determine the interest rate on bonds and capital income taxes. Note that, in reality, only one of these variables is required: given that r_t is known, if we manage to pin down θ_t, then we get R_t^k and R_t^b can be extracted from a non-arbitrage condition. From the Euler Equation for capital, replacing for p_t:

$$\pi_t(s^t)u_c(s^t) = \sum_{s_{t+1}|s^t} \beta\pi_{t+1}(s^{t+1})u_c(s^{t+1})R_{t+1}^k(s^{t+1})$$

From the Euler Equation for bonds

$$\pi_t(s^t)u_c(s^t) = \sum_{s_{t+1}|s^t} \beta\pi_{t+1}(s^{t+1})u_c(s^{t+1})R_{t+1}^b(s^{t+1})$$

and the budget constraint at $t+1$:

$$\begin{gathered} c_{t+1}(s^{t+1}) + k_{t+1}(s^{t+1}) + b_{t+1}(s^{t+1}) \\ = (1-\tau_{t+1}(s^{t+1}))w_{t+1}(s^{t+1})l_{t+1}(s^{t+1}) + R_{t+1}^k(s^{t+1})k_t(s^t) + R_{t+1}^b(s^{t+1})b_t(s^t) \end{gathered}$$

we obtain $n+2$ equations on $2n$ unknowns for the return rates, assuming that there are n nodes in the immediate succession of s^t. There are $2n$ unknown return rates: one for capital and one for bonds at each successor node. There are $2+n$ equations: 2 Euler Equations and n budget constraints at $t+1$. Note that at this point we have already determined **everything** else that enters the budget constraints and Euler Equations, except for the rates of return! Clearly, we can only retrieve the rates of return when $n = \{1, 2\}$. There is an indeterminacy that cannot be solved without further assumptions on the model. A way to solve this problem is to assume that $\theta_t(s^t)$ is constant *over successor nodes*. That is, the capital income tax rate may differ across time, but it does not differ across nodes for any given point in time. In case this assumption is made, recall that

$$R_{t+1}^k(s^{t+1}) = 1 + (1-\theta_{t+1})[r_{t+1}(s^{t+1}) - \delta]$$

and replace this in the Euler Equation for capital. Clearly, we can now solve for the unique θ_{t+1}:

$$\pi_t(s^t)u_c(s^t) = \sum_{s_{t+1}|s^t} \beta\pi_{t+1}(s^{t+1})u_c(s^{t+1})[1 + (1 - \theta_{t+1})[r_{t+1}(s^{t+1}) - \delta]]$$

Thus we can then construct the sequence of bond returns and the equilibrium is fully characterised. □

4.4.1 Indeterminacy and Tax Regime Equivalence

It is immediate to see that, using the above proposition, the same competitive equilibrium allocation can be reproduced using a myriad of different tax instruments. The proof in the previous section departs from a particular equilibrium allocation and constructs a Ramsey equilibrium based only on two tax instruments: labour and capital income taxes. In principle, the government could have an arbitrary number of tax instruments at its disposal and still be able to induce the exact same competitive equilibrium allocation.

The degree of indeterminacy, however, is greater than what the above paragraph suggests. In the last part of the proof of equivalence of the primal approach, we are faced with an indeterminacy arising on the capital income tax. This should make it clear that *even if two fiscal authorities are endowed with the same instruments, different menus of policies are compatible with the same competitive equilibrium allocation.* In other words, the same real allocations can be consistent with different sequences of tax rates, even if the tax instruments themselves are the same. This is because, thanks to its nature, the Ramsey problem identifies *wedges* and distortions: allocation differences that are generated by the fact that some instruments are distortionary. It does not determine, directly, tax instruments, but rather allocations.

To see this more clearly, consider an economy whose fiscal authority has three tax instruments at its disposal: the previous two (τ_t, θ_t) as well as a consumption tax τ_t^c. In this case, the intra and intertemporal optimality conditions for the agent can be easily checked to be of the type

$$-\frac{u_l(s^t)}{u_c(s^t)} = \frac{1 - \tau_t}{1 + \tau_t^c} w_t(s^t)$$

$$\frac{u_c(s^t)}{1 + \tau_t^c} = \sum_{s_{t+1}|s^t} \beta\pi_{t+1}(s_{t+1})\frac{u_c(s^{t+1})}{1 + \tau_{t+1}^c}[1 + (1 - \theta_{t+1})[r_{t+1}(s^{t+1}) - \delta]]$$

Consider an alternative authority which, while endowed with the same set of instruments ϕ, chooses different tax levels $\hat{\phi}$. In particular, it sets the capital income tax to zero, $\hat{\theta}_t = 0, \forall t \geq 0$. Thus the Euler Equation, in this economy, is

$$\frac{u_c(s^t)}{1 + \hat{\tau}_t^c} = \sum_{s_{t+1}|s^t} \beta\pi_{t+1}(s_{t+1})\frac{u_c(s^{t+1})}{1 + \hat{\tau}_{t+1}^c}[1 - \delta + r_{t+1}(s^{t+1})]$$

It is clear that, given the same initial conditions, if ϕ and $\hat{\phi}$ are such that

$$\frac{1+\tau_t^c}{1+\tau_{t+1}^c}[1+(1-\theta_{t+1})[r_{t+1}(s^{t+1})-\delta]] = \frac{1+\hat{\tau}_t^c}{1+\hat{\tau}_{t+1}^c}[1+r_{t+1}(s^{t+1})-\delta]$$

and

$$\frac{1-\tau_t}{1+\tau_t^c} = \frac{1-\hat{\tau}_t}{1+\hat{\tau}_t^c}$$

then the consumer's decisions are all distorted in exactly the same way, for any period and any node. Therefore, these two different tax policies will generate exactly the same competitive equilibrium. This highlights the fact that what matters for real allocations (and, therefore, welfare) are wedges and distortions, and not the taxes directly.

Naturally, if more and more tax instruments are added, the dimension of the indeterminacy will grow.

4.4.2 Ramsey Taxation in Practice

At this point, the following proposition should contain an obvious result:

Proposition 4.4.2. *The allocations in a Ramsey Equilibrium solve the following problem*

$$\max_{\{c_t(s^t),l_t(s^t),k_t(s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t(s^t) u[c_t(s^t), l_t(s^t)]$$

subject to the Feasibility and Implementability constraints, for (k_{-1}, b_{-1}) *given.*

So that, as suggested in previous sections, it is usually easier to adopt the Primal Approach, find the optimal allocations and then back out the equilibrium, than to solve for optimal taxes subject to the full equilibrium conditions.

To see an application of this, let λ denote the Lagrange multiplier on the Implementability constraint (there is a unique constraint of this type) and $\{\mu_t\}$ be the sequence of Lagrange multipliers on the feasibility constraints for each period. It is useful to define the following function (I omit state dependence)

$$W(c_t, l_t, \lambda) = u(c_t, l_t) + \lambda[u_c c_t + u_l l_t]$$

where, note, it contains the 'summable' part of the implementability constraint, with the multiplier already included. Then, the Lagrangian for the Primal Problem can be written as

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t W(c_t, l_t, \lambda) - \lambda u_c(s_0)[R_0^k k_{-1} + R_0^b b_{-1}] \\ &+ \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t \mu_t [F(k_{t-1}, l_t, s_t) + (1-\delta)k_{t-1} - c_t - g_t - k_t]\end{aligned}$$

The typical FOC for a period $t \geq 1$ will be of the type

$$-\frac{W_l(s^t)}{W_c(s^t)} = F_l(s^t)$$
$$\pi_t W_c(s^t) = \sum_{s_{t+1}|s^t} \beta \pi_{t+1} W_c(s^{t+1})[1 + F_k(s^{t+1}) - \delta]$$

which are extremely similar to the FOC we would find in a regular representative problem. The only difference is that utility has been replaced by the *distorted* utility function W. This function W is equal to utility, period by period, plus a term that accounts for the distortions generated by taxes over equilibrium allocations, the 'summable' part of the implementability constraint. Note that each of its derivatives, W_c and W_l will usually depend on second-derivatives and cross-derivatives, given that the component of the implementability constraint that it includes incorporates marginal utility terms. For example, the first FOC will be

$$-\frac{u_l(s^t) + \lambda u_{cl}(s^t)c_t + \lambda u_l(s^t) + \lambda u_{ll}(s^t)l_t}{u_c(s^t) + \lambda u_{cc}(s^t)c_t + \lambda u_c(s^t) + \lambda u_{lc}(s^t)l_t} = F_l(s^t)$$

In spite of this, we can obtain some immediate insights from the above optimality conditions regarding optimal taxation. To understand this, consider the second FOC, and impose a non-stochastic steady state. Then, it tells us that

$$1 = \beta(1 + F_k - \delta)$$

which is exactly the same optimality result that we would obtain in a regular growth model with no government, the modified golden-rule. Note that this implies, immediately, that capital taxes should be equal to zero, at least in the long-run. If they were non-zero, the above condition would feature a wedge term on the marginal productivity of capital. This is a well-known result derived by Chamley (1986).

The intuition behind this is that capital taxation distorts savings decisions, and induces agents to underaccumulate capital and underconsume throughout their lives. Not only that, but it also forces them to work more. When the government only employs distortionary tax instruments, such as labour income and consumption taxes, the distortions generated by these are amplified in terms of welfare losses.

To understand how perverse capital taxation is, consider the case of some economy A where capital income is taxed. For simplicity, let us focus on the non-stochastic case. The Euler Equation in that economy is then

$$\frac{u_{c,t}}{u_{c,t+1}} = \beta[1 + (1 - \theta_{t+1})[F_{k,t+1} - \delta]]$$

Now, consider some other economy B, where a consumption tax is levied instead of a capital tax. The Euler Equation is thus

$$\frac{u_{c,t}}{u_{c,t+1}} = \frac{1 + \tau_t^c}{1 + \tau_{t+1}^c} \beta[1 + F_{k,t+1} - \delta]$$

Clearly, for real allocations to coincide in these two economies, given the same initial conditions, we must have that

$$1 + (1 - \theta_{t+1})[F_{k,t+1} - \delta] = \frac{1 + \tau_t^c}{1 + \tau_{t+1}^c}[1 + F_{k,t+1} - \delta]$$

or

$$\frac{1 + \tau_t^c}{1 + \tau_{t+1}^c} = \frac{1 + (1 - \theta_{t+1})[F_{k,t+1} - \delta]}{[1 + F_{k,t+1} - \delta]}$$

Assuming the allocations coincide, and that the marginal productivity of capital is thus identical, we must have that from $\theta_{t+1} > 0$

$$\frac{1 + \tau_t^c}{1 + \tau_{t+1}^c} < 1$$

Thus, capital income taxes amount to *ever increasing* taxes on consumption. That is the only way for the two economies to have the same allocation. Thus capital income taxes are an ideal way for an economy to be led to total obliteration in the long-run.